**Motion of gyroscopes and spinning tops**

There is a gyroscope inside almost all smartphones [Figure 1]. Your iPhone and other smartphones all use a gyroscope inside them to continuously monitor your motion. Using the internal gyroscope, the smartphone can tell in which direction you are walking (e.g., 37.2 degrees Northwest) and continuously monitor how your altitude changes (e.g., walking up the stairs to reach a height of 2.67m). This continuous, real-time, and precise monitoring of your motion is due to a gyroscope. Despite its use in such modern devices, the gyroscope is not a new invention. It is actually a device that was invented many centuries ago. In this lecture, we will deduce how it works.

**Fig. 1. The gyroscope:** (A) Basic parts of a gyroscope (image from [https://en.wikipedia.org/wiki/Gyroscope](https://en.wikipedia.org/wiki/Gyroscope)). (B) There is a gyroscope inside your iPhone (and other smartphones). By rotating your phone around the three perpendicular axes, you can control the yaw, pitch, and roll of the gyroscope inside your phone. Your phone measures how the gyroscope responds to your rotations. It then detects how you're moving on a map app. For example, the phone can tell which direction you're turning as you're walking and which direction is North or West. (image from [http://www.mathworks.com](http://www.mathworks.com)). (C & D) Some typical motions of the gyroscope that we will learn in this lecture (image from [http://hyperphysics.phy-astr.gsu.edu/hbase/gyr.html](http://hyperphysics.phy-astr.gsu.edu/hbase/gyr.html)).
To understand gyroscopes, let's first analyze the motion of a spinning top [Figure 2]. Spinning tops are perhaps more familiar to you than gyroscopes. Gyroscopes and spinning tops work in the same way. Thus once we understand how spinning tops work, we will understand how gyroscopes work.

A spinning top has an angular momentum \( \vec{L} \) [Figure 2]. The rate of change of this angular momentum with respect to time (i.e., \( d\vec{L}/dt \)) is the torque \( \vec{\tau} \) that acts on the spinning top. The torque is due to the gravity that acts downwards (towards the -z direction in Fig. 2) at the spinning top's center of mass. So it exerts a torque about the pivot point (i.e., the point of contact between the ground and the spinning top, this is the origin (0,0,0) of the coordinate system in Figure 2). The torque is

\[
\vec{\tau} = \vec{r} \times \vec{F}
\]

Let's first calculate the magnitude of the torque while ignoring the direction of the torque vector. The cross product of the two vectors, \( \vec{r} \) and \( \vec{F} \), gives us

\[
|\vec{\tau}| = Mg \cdot r \cdot \sin \theta
\]

\( \theta \) is the angle between \( \vec{r} \) and \( \vec{F} \) when you slide the vector \( \vec{F} \) so that its tail is also at the origin just like the tail of vector \( \vec{r} \). Now let's deduce the direction of the torque vector. We use the right hand rule to take the cross product. This means that you stretch your right hand and...
point it parallel to and in the same direction as \( \vec{r} \). Then you curl all of your right hand's fingers except your right thumb towards the vector \( \vec{F} \) to form a fist without your thumb. Your thumb now points in the direction of \( \vec{r} = \vec{r} \times \vec{F} \). Let's call this vector \( \vec{\phi} \). The "hat" symbol above this Greek letter "phi" \( \phi \) means that \( \vec{\phi} \) is a unit vector (i.e., a vector with length 1). So it is vector of length 1 that points in the same direction as your thumb (the direction of \( \vec{r} \)) [Note: If you're uncomfortable with a unit vector, just remember that you already worked with it before. Other examples of unit vectors are: \( \vec{x}, \vec{y}, \vec{z} \) (the unit vectors pointing in the x, y, and z directions respectively. Sometimes they are also written as \( \vec{i}, \vec{j}, \vec{k} \)).

Now since
\[
\frac{d\vec{L}}{dt} = \vec{r}
\]
we have
\[
\frac{d\vec{L}}{dt} = (Mg \gamma \sin \theta)\vec{\phi}
\]
So at a given instant in time, the angular momentum of the spinning top is changing in the direction of \( \vec{\phi} \). This means that after a small (infinitesimal) time \( dt \) later, the angular momentum of the spinning top points in a new direction [Fig. 3].

To calculate this new direction, we multiply both sides of above equation by \( dt \) to get
\[
d\vec{L} = \vec{r} dt
\]
\( d\vec{L} \) is the small (infinitesimal) vector by which \( \vec{L} \) changes in time interval \( dt \). It is infinitesimal because \( dt \) is very small. \( d\vec{L} \) is the change in the angular momentum \( \vec{L} \) due to the torque. Specifically, it is the change in the angular momentum from time \( t \) to \( t+dt \). We write this mathematically as
\[
\vec{L}(t+dt) = \vec{L}(t) + \vec{r} dt
\]
Now, consider an infinitesimal time after \( t \). We are now at time \( t+dt \). What is the torque now?
The force of gravity is still the same. So the force vector \( \vec{F} \) is still pointing straight down to the ground (in the -z direction, Fig. 2). Note also that the position vector \( \vec{r} \) still starts from the point of contact (the origin \((0,0,0)\)) and its tip ends at the position of the center of mass of the spinning top.
Fig. 3. Analysis of precession of the spinning top: (A) We decompose the angular momentum into two vectors that are perpendicular to each other: (1) $\vec{L}_{xy}$ lies entirely on the xy-plane, and (2) $\vec{L}_z$ is parallel to the z-axis. $\vec{L} = \vec{L}_{xy} + \vec{L}_z$. (B) Since the torque vector lies in xy-plane (i.e. has no z-component), $\vec{L}_z$ does not change over time. Only the $\vec{L}_{xy}$ changes over time. Since the torque vector is perpendicular to $\vec{L}_{xy}$, the length (magnitude) of the $\vec{L}_{xy}$ does not change over time only its direction changes. The tip (arrowhead) of the vector $\vec{L}_{xy}$ traces out a circle as shown. (C) The key idea here is that for an infinitesimal (very tiny) time interval $dt$, the arc length (shown in orange) is the same as the length of the vector $d\vec{L}$ (we write $|d\vec{L}|$ to represent the length of the vector). This length is $L_{xy}d\phi$. Here the $L_{xy}$ represents the length of the vector $\vec{L}_{xy}$ (i.e., $|\vec{L}_{xy}| = L_{xy}$).
So now we have exactly the same situation as the situation we had at time \( t \). So we still have

\[
\frac{d\vec{L}}{dt} = (M g r \sin \theta) \hat{\phi}
\]

Here \( \hat{\phi} \) is still the unit vector that is perpendicular to both \( \vec{r} \) and \( \vec{F} \), just like it was at time \( t \). But since the vector \( \vec{L} \) has changed its direction from time \( t \) to time \( t+dt \), the direction of \( \hat{\phi} \) at time \( t \) and its direction at time \( t+dt \) are not the same. \( \hat{\phi} \) is also changing over time. At the previous time \( t \), \( \hat{\phi} \) was perpendicular to both \( \vec{r} \) and \( \vec{F} \) at that time. At the later time \( t+dt \), \( \hat{\phi} \) is perpendicular to \( \vec{r} \) and \( \vec{F} \) at this new time \( t+dt \). Thus again, repeating the previous argument, we now have

\[
\vec{L}(t + dt + dt) = \vec{L}(t + dt) + \vec{r} dt
\]

Now what happens at another time step \( dt \) later? The vectors \( \vec{r} \) and \( \vec{F} \) at time \( t+dt+dt \) would still be described by the same properties as before. So we just repeat the above argument again. And then even a further time step \( dt \) later, we would repeat above reasoning again. And then for the time step \( dt \) thereafter, we would repeat the same argument. Another time step later, we repeat yet again the same argument and so on and so forth. If we draw the path of the vector \( \vec{L} \) over a many time steps, by piecing together all the different \( dt \) time steps together, we would find the complete path that \( \vec{L} \) follows over all time. We note two facts about this path. First, the tail-end of the vector \( \vec{L} \) (where the vector begins), is always at the point of contact (where the spinning top is pivoted - the origin \((0,0,0))\). Secondly, the tip (arrowhead) of the vector \( \vec{L} \) traces out a circle over time. Note that this is like the centripetal motion that we studied a few weeks ago. There we saw that for a particle moving at a constant speed (but not constant velocity) in a circle, its velocity \( \vec{v} \) (and thus its linear momentum vector \( \vec{p} \)) was always perpendicular to the force vector \( \vec{F} \) at all times. We saw there that the magnitude of \( \vec{p} \) was constant at all times and the magnitude of the force vector \( \vec{F} \) (which we called the "centripetal force" because it always pointed to the center of the circle) remained constant at all times. Likewise, we have a similar situation for the spinning top. The circular rotation of the spinning top's axis of rotation (and thus its vector \( \vec{L} \)) is called \textbf{precession}. We say that the spinning top's axis of rotation \textbf{precesses} in a circle while the top is spinning.

What is the angular speed of the precession \( \omega_p \) around z-axis [Fig. 2]? Does the angular velocity \( \vec{\omega} \) of the spinning motion change over time? To answer this question, note that we can break the vector \( \vec{L} \) into two components: One that lies entirely in the xy-plane (call it \( \vec{L}_{xy} \)) and one that is parallel to the z-axis (call it \( \vec{L}_z \)) [Figure 3]. Since the torque vector \( \vec{\tau} \) lies in the xy-plane and does not have a z-component, we can see that the \( \vec{L}_z \) does not
change over time. It remains constant in direction and length. Only the vector \( \vec{L}_{xy} \) changes over time and its rate of change over time is the torque. In other words,

\[
\frac{d\vec{L}}{dt} = \frac{d\vec{L}_{xy}}{dt} = \vec{\tau}
\]

Let \( \phi \) be the angle that \( \vec{L}_{xy} \) sweeps out in the xy-plane. Then we have

\[
dL = L_{xy} d\phi
\]

Here we use \( L_{xy} \) to denote the magnitude (length) of the vector \( \vec{L}_{xy} \). That is, \( L_{xy} = |\vec{L}_{xy}| \).

Similarly, \( dL = |d\vec{L}| \). Dividing both sides by \( dt \), we get

\[
\frac{dL}{dt} = L_{xy} \frac{d\phi}{dt}
\]

This becomes

\[
\tau = L_{xy} \omega_p
\]

since \( \omega_p = \frac{d\phi}{dt} \) by definition. Here again, \( \tau \) (without the arrow above it) is the length of the torque vector \( \vec{\tau} \). Noticing that \( L_{xy} = L \sin \theta \), we get

\[
\tau = L \sin \theta \omega_p
\]

So

\[
\omega_p = \frac{\tau}{L \sin \theta}
\]

Plugging in \( |\vec{\tau}| = Mg r \) and \( |\vec{L}| = I |\vec{\omega}| \), we get

\[
\omega_p = \frac{Mgr}{I |\vec{\omega}|}
\]

This is the famous angular speed of precession for spinning tops and gyroscopes. Note that like the angular momentum \( \vec{L} \), the angular velocity \( \vec{\omega} \) of the spin precesses around the z-axis together with \( \vec{L} \). Just like the magnitude of \( \vec{L} \) (i.e., the \( |\vec{L}| \)), the magnitude of \( \vec{\omega} \) (i.e., the \( |\vec{\omega}| \)) remains constant over time.

Note that the faster the top spins (i.e., the larger the \( |\vec{\omega}| \) is), the slower the top precesses around the z-axis (i.e., the smaller the \( \omega_p \) is). The slower the top spins, the faster the top precesses around the z-axis. We can also compute the linear speed of the precession, instead of just the angular precession speed. Simply noting that the center of mass is a distance \( r \sin \theta \) away from the z-axis, we note that the linear speed of precession \( v_p \) is

\[
v_p = r \sin \theta \omega_p = \frac{Mgr^2}{I |\vec{\omega}|} \sin \theta
\]
The \( \sin \theta \) term that appears in \( v_p \), but not in \( \omega_p \), is simply saying that the more tilted the top is (i.e. as the \( \theta \) goes from 0 to \( \pi / 2 \) radians), the CM of the top sweeps out more length (more arc length of the circle) per unit time for the same angular speed of precession.

Now that we understand how a spinning top precesses, let's get back to our original starting point - the gyroscope. How does a smartphone tell how you're moving with its internal gyroscope? Imagine a gyroscope whose rotor (see Fig. 1A) is spinning with a constant angular speed \( \omega \) about its "spin axis" (see Fig. 1A). It is precessing with some angular velocity \( \omega_p' \). The spin axis is attached to a metal frame called the "gimbal" (see Fig. 1A). The gimbal is attached to a larger frame called the gyroscope frame through hinges. The hinges are well oiled so they act as frictionless bearings. This means that you can tilt the gyroscope frame (Fig. 1D), but since the hinges cannot exert force on the gimbal and the spinning axis (due to lack of friction), the gimbal and the spinning wheel will maintain their position (compare the picture in Figs. 1C and 1D). For example, if the angular momentum was horizontal and the center of mass of the spinning wheel was directly above the gyroscope's support stand (\( \theta = 0 \)) before rotating the gyroscope frame (Fig. 1C), it will remain horizontal after rotating the frame (Fig. 1D). But now, there is torque acting on the wheel because its center of mass is no longer directly above the gyroscope's support stand (i.e. \( \theta \) is no longer zero) (Fig. 1D). This causes the gyroscope to precess. Your phone measures the angular speed and direction of this precession very precisely. From this, it can precisely and continuously monitor by how much you have flipped your phone (and which direction you're walking on a map).