Mon., July 14, 08

Last class, we found that

$$f_-(x,t) = A\sin(kx-\omega t) \qquad \text{(right moving sinusoidal wave)}$$

and $$f_+(x,t) = B\sin(-kx-\omega t) \qquad \text{(Left moving sinusoidal wave)}$$

are both 2 particular solutions to the wave eq'n.

It's not hard to convince yourself (do it yourself!) that

$$h_-(x,t) = C\cos(kx-\omega t) \qquad \text{(right moving)}$$

and $h_+(x,t) = D\cos(-kx-\omega t)$ (left moving) are both solutions to the wave eq'n as well.

And by linearity of wave eq'n we showed on previous pages, we know that

Sum of solutions is also a solution to the wave eq'n.

so: $$f(x,t) \equiv f_-(x,t) + f_+(x,t) + h_-(x,t) + h_+(x,t) \quad \text{is also a solution.}$$

$$= A\sin(kx-\omega t) + B\sin(-kx-\omega t) + C\cos(kx-\omega t) + D\cos(-kx-\omega t).$$

(A, B, C, D are arbitrary constants.)

Notice how long and cumbersome it is to write this out.

Remedy to our "Cumbersome" problem: Use $\mathbb{C}$ numbers:

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2} \quad \text{(original wave eq'n)} \xrightarrow[f \to \tilde{f}]{} \frac{\partial^2 \tilde{f}}{\partial t^2} = v^2 \frac{\partial^2 \tilde{f}}{\partial x^2}$$

($\mathbb{R}$: physical problem) $\longrightarrow$ ($\mathbb{C}$: not real, math (not physics) problem.)

Here, all we've done is allow $f(x,t)$ to now take on $\mathbb{C}$-values and call it $\tilde{f}(x,t)$. Our strategy is, solve the $\mathbb{C}$-valued equation, then extract a real sol'n $f(x,t)$ out of $\tilde{f}(x,t)$.

This is beneficial since $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ ← Addition of sine & cosine, but written in a compact form.

, Also, linearity is really what makes the $\mathbb{C}$-equivalent wave eq'n so useful to us since:

$$\tilde{f}(x,t) = \underbrace{\tilde{f}_{Re}(x,t)}_{\text{↑ real}} + \underbrace{i\,\tilde{f}_{Im}(x,t)}_{\text{↑ Imaginary Part}} \quad \text{(but } \tilde{f}_{Im}(x,t) \text{ is } \underline{\text{real}}.)$$

~~[illegible]~~ and if $\tilde{f}(x,t)$ is a solution to wave eq'n then:

$$\frac{\partial^2 \tilde{f}}{\partial t^2} = v^2 \frac{\partial^2 \tilde{f}}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left(\tilde{f}_{Re} + i\tilde{f}_{Im}\right) = v^2 \frac{\partial^2}{\partial x^2}\left(\tilde{f}_{Re} + i\tilde{f}_{Im}\right)$$

$$\Rightarrow \underbrace{\left(\frac{\partial^2 \tilde{f}_{Re}}{\partial t^2} - v^2 \frac{\partial^2 \tilde{f}_{Re}}{\partial x^2}\right)}_{\text{↑ real \# ①}} = i \underbrace{\left(v^2 \frac{\partial^2 \tilde{f}_{Im}}{\partial x^2} - \frac{\partial^2 \tilde{f}_{Im}}{\partial t^2}\right)}_{\text{↑ real \# ②}}$$

The only way that above eq'n can be satisfied is if ① = 0 and ② = 0

That is: $\dfrac{\partial^2 \tilde{f}_{Re}}{\partial t^2} = v^2 \dfrac{\partial^2 \tilde{f}_{Re}}{\partial x^2}$

and $\dfrac{\partial^2 \tilde{f}_{Im}}{\partial t^2} = v^2 \dfrac{\partial^2 \tilde{f}_{Im}}{\partial x^2}$

both $\tilde{f}_{Re}(x,t)$ and $\tilde{f}_{Im}(x,t)$ are ~~solution~~ each solutions to the real wave eq'n (since both $\tilde{f}_{Re}$ and $\tilde{f}_{Im}$ are real-valued functions.)

~~So, our strategy from now~~

Hence, we've really found 2 real solutions $\tilde{f}_{Re}$ & $\tilde{f}_{Im}$ just by ~~working with~~ ~~one~~ ~~$\mathbb{C}$-valued~~ finding one $\mathbb{C}$-valued solution $\tilde{f}(x,t)$.

Less work $\Longrightarrow$ happiness!!

So, from now on, we will solve the $\mathbb{C}$-valued wave eq'n: $\dfrac{\partial^2 \tilde{f}}{\partial t^2} = v^2 \dfrac{\partial^2 \tilde{f}}{\partial x^2}$

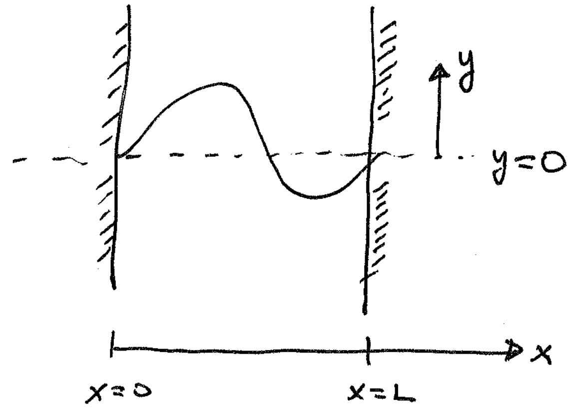
find $\tilde{f}(x,t) = \tilde{f}_{Re}(x,t) + i\tilde{f}_{Im}(x,t)$, then say that the real solutions are $\tilde{f}_{Re}(x,t)$ and $\tilde{f}_{Im}(x,t)$.

(And in fact: $\boxed{f(x,t) = A\tilde{f}_{Re}(x,t) + B\tilde{f}_{Im}(x,t)}$)

A & B constants. (real)

their sum is a solution too, by linearity obeyed by wave eq'n.)

Ex



A string with its 2 ends fixed at 2 walls: $(x=0, x=L)$

$\Rightarrow$ $y(x=0,t)=0$
$y(x=L,t)=0$ at all time t.

Hence, the $\mathbb{C}$-equivalent $\tilde{y}(x,t)$ must also obey:

$\tilde{y}(x=0,t)=0$
and $\tilde{y}(x=L,t)=0$ at all time t, as well.

Now, we know that $\tilde{y}(x,t) = [Ae^{ikx} + Be^{-ikx}]e^{-i\omega t}$

is a sol'n to $\mathbb{C}$-valued wave eq'n: $\dfrac{\partial^2\tilde{y}}{\partial t^2} = v^2 \dfrac{\partial^2 \tilde{y}}{\partial x^2}$.

But, now, we have additional requirement that:

$\tilde{y}(x=0,t)=0=[A+B]e^{-i\omega t}$ $\Rightarrow$ $A+B=0$ $\Rightarrow$ $A=-B$

And so $\tilde{y}(x,t) = A[e^{ikx} - e^{-ikx}]e^{-i\omega t}$

$= 2iA\sin(kx)\,e^{-i\omega t}$

Now, one more requirement:

$\tilde{y}(x=L,t)=0 = e^{-i\omega t}\,2iA\sin(kL)$

But $A \neq 0$ since if it is, then $\tilde{y}(x,t)=0$. ← trivial (uninteresting solution).

$\Rightarrow$ $\sin(kL)=0$.

$\Rightarrow$ $kL = n\pi$ $n=0, \pm1, \pm2, \pm3, \ldots$

$\Rightarrow$ $k_n = n\pi/L$.

$\therefore$ $\tilde{y}(x,t) = e^{-i\omega t}\,2iA\sin\left(\dfrac{n\pi x}{L}\right)$. ; but we don't want $n=0$ (trivial solution).

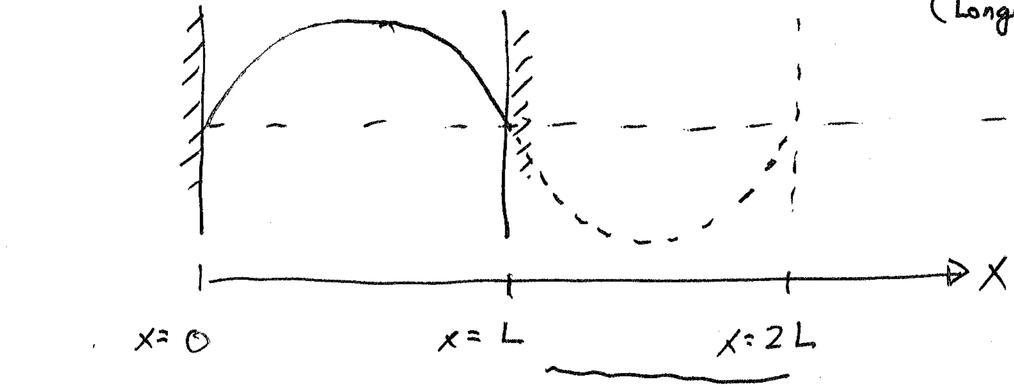
~~and we can~~

so $n=\pm1, \pm2, \ldots$

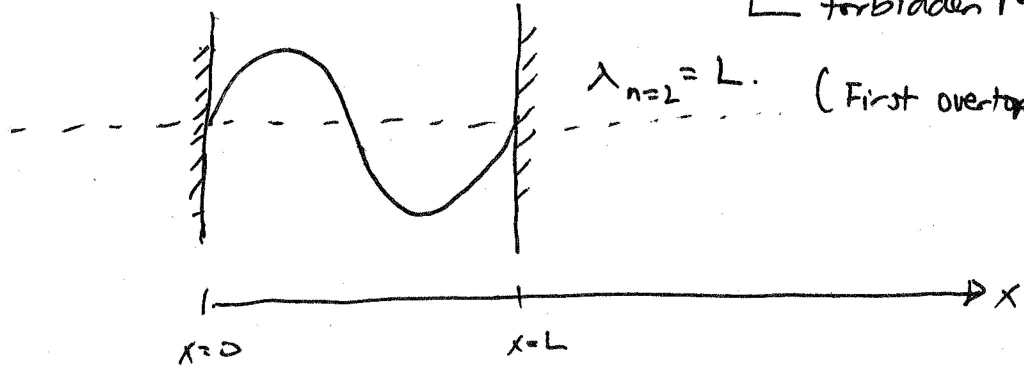
over

~~And~~, so: $k_n = \frac{2\pi}{\lambda_n}$: $\lambda_{n=1} = 2L$ (Fundamental mode $n=1$.)

(Longest wavelength possible)



forbidden region (beyond wall.)



$\lambda_{n=2} = L$. (First overtone : $n=2$)

and so on.

What about ω? Does it also take on only certain values?

Ans : Yes: $\omega_n = k_n v$

$$\Rightarrow \boxed{\omega_n = \frac{n\pi v}{L}} \qquad n = 1, 2, 3 \ldots\ldots$$

$\Rightarrow$ Longer the wavelength (smaller n), the shorter the frequency $\frac{\omega_n}{2\pi} = f_n$. $\Rightarrow$ Lower the pitch of guitar string.

ω_n ← called "normal mode angular frequencies"

Furthermore, if $n<0$, we can always take the $\ominus$ sign outside of the sine, absorb it into the constant A:

i.e. $$2iAe^{-i\omega t}\sin\left(\frac{n\pi x}{L}\right) \underset{(n<0)}{=} 2iAe^{-i\omega t}(-1)\sin\left(\frac{|n|\pi x}{L}\right)$$

$$= \underbrace{2i(-A)\,e^{-i\omega t}\sin\left(\frac{|n|\pi x}{L}\right)}.$$

Just the sol'n with $n>0$. but now pick "A" to be "$-A$".

So, $n<0$ cases are redundant.

$\therefore$ $\mathbb{C}$-equivalent sol'n is:

$$\boxed{\tilde{y}_n(x,t) = 2i\tilde{A}_n e^{-i\omega_n t}\sin\left(\frac{n\pi x}{L}\right)} \qquad n=1,2,3\ldots$$

where $\omega_n =$

$$= -2i\tilde{A}_n\left[\cos(\omega_n t) + i\sin(\omega_n t)\right]\sin\left(\frac{n\pi x}{L}\right).$$

So, the ~~[illegible]~~ real, physically meaningful solution $y_n(x,t)$ is:

$$\boxed{y_n(x,t) = \left(A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right)\sin\left(\frac{n\pi x}{L}\right)} \quad (*)$$

where, $$\boxed{(A_n^2 + B_n^2)^{1/2} = 2\tilde{A}_n}$$

summary: $\boxed{k_n = \frac{n\pi}{L}} \Rightarrow \boxed{\lambda_n = \frac{2\pi L}{n\pi} = \frac{2L}{n}}$, and $\boxed{\omega_n = \frac{n\pi v}{L}}$

$n=1,2,3\ldots\ldots$

rg96

In the problem we just looked at, we saw that "k" cannot take on just any value. Also, the constants A & B in $Ae^{ikx} + Be^{-ikx}$ cannot take on any value. ($\because A = -B$)

(so A & B are not independent of each other.)

All these restrictions came about because of boundary conditions.

(2 boundary conditions:
$y(x=0, t) = 0$
and $y(x=L, t) = 0$)

[This happens almost all the time in physics: what is physically realizable state of a system is determined by restrictions placed on the system by boundary conditions.]

Introduction to Fourier series:

Going back to the previous example of standing waves between 2 walls, we found that $y_n(x,t) = (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) \sin(\frac{n\pi x}{L})$ is a solution for $n = 1, 2, 3, \ldots$.

These are called normal modes.
(i.e. Each value of n gives one normal mode y_n.)

But by linearity, (superposition principle):

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) \sin\left(\frac{n\pi x}{L}\right).$$

This is also a solution, where A_n, B_n can be any arbitrary values.

OK, but are there solutions $y(x,t)$ that <u>cannot</u> be expressed as a sum of the y_n's ?

<u>Ans</u>: NO! The remarkable fact is that practically <u>all</u> physically realizable motion of string fixed between the 2 walls can be expressed as a sum of the normal modes!

That is: $$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi V}{L}t\right) + B_n \sin\left(\frac{n\pi V}{L}t\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

By picking the right values of A_n and B_n for each n, any physically realizable waveform $y(x,t)$ between the 2 walls can be expressed as the sum written above.

* <u>Why is this so?</u> Unfortunately, a rigorous proof of this fact is both time consuming and require techniques we don't yet have in our arsenal. But we can try to get an intuitive understanding of why this ~~must~~ be so, based on what we've done so ~~far~~. (see pg 74 & 75 of lecture notes):

Recall that when we looked at coupled oscillations of few particles, (2 or 3), we found out their normal modes

(# normal modes = # of particles in the coupled oscillator.)

Looking back at those (see pg 73, 74, 75 of lecture notes), we guessed solution of N coupled oscillators (typically, we had $N = 2$ or 3)

to be $$\begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_N(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} e^{-i\omega t}$$ ← <u>Normal mode guess sol'n</u>

And we found from the matrix EOM that ω was normal mode angular frequency.

Pg98

Looking back at how we ~~solved~~ ~~the~~ found the standing waves on Pg93;

we guessed $\tilde{y}(x,t) = g(x) \cdot e^{-i\omega t}$

where $g(x) = Ae^{ikx} + Be^{-ikx}$;

But remember, at each x value, there's a little particle ("atom") making up part of the string (or slinky); After all, this was how we derived the wave eqn. (and $x = ja$)

← jth particle in slinky example on Pg 76.

so, this is like saying
$$\begin{pmatrix} Z_1(t) \\ Z_2(t) \\ \vdots \\ Z_N(t) \end{pmatrix} = \begin{pmatrix} \tilde{y}(x_1,t) \\ \tilde{y}(x_2,t) \\ \vdots \\ \tilde{y}(x_N,t) \end{pmatrix} = \begin{pmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_N) \end{pmatrix} e^{-i\omega t}$$

just as we did when we looked for normal modes when N=2, or 3. Except that now, N is much larger. This is why the ω_n's we found for standing wave on Pg95 are called "normal mode angular frequencies"

Now, going back to justifying why any $y(x,t)$ can be expressed:

$$y(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi v t}{L}\right) + B_n \sin\left(\frac{n\pi v}{L} t\right)\right) \sin\left(\frac{n\pi x}{L}\right)$$

Just as when we had N = 2, 3 ~~coup~~ particle coupled oscillators, we found out that when we added up all the normal modes (2, if 2 particles), (3, if 3 particles), then we ended up getting just the right number of free parameters.

In much the same way, when N is very large, in fact, N = ∞, adding up the normal modes $y_n(x,t)$ gives the right # of free parameters.

$$y(x,t) = \sum_{n=1}^{\infty} \left(\underline{A_n} \cos\left(\frac{n\pi v t}{L}\right) + \underline{B_n} \sin\left(\frac{n\pi v}{L} t\right)\right) \sin\left(\frac{n\pi x}{L}\right).$$

2 free parameters for each n

$\Rightarrow$ 2N free parameters for general solution $y(x,t)$

(Just as we had $2 \cdot 2 = 4$ ($N=2$ particles in coupled oscillator)

$2 \cdot 3 = 6$ ($N=3$ particles in coupled oscillator).

Although this was not a rigorous mathematical proof, I hope it convinced you, at least intuitively, why the Fourier series works.

So, the general solution to the wave eq'n with fixed boundaries (between 2 walls separated by length L)

is:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi v t}{L}\right) + B_n \sin\left(\frac{n\pi v}{L} t\right)\right) \sin\left(\frac{n\pi x}{L}\right)}$$

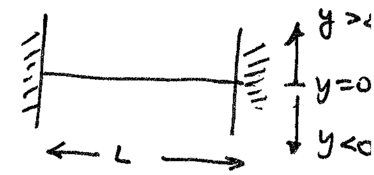
. Our goal now is to find what A_n and B_n must be, and how we find those values in specific conditions.

Let's do that next.

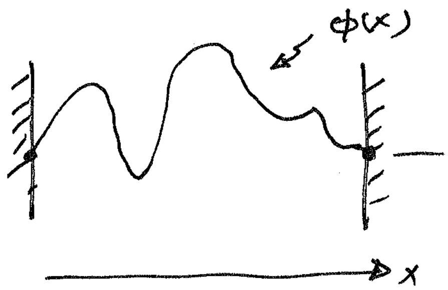
over

Ex: Finding A_n & B_n :

Suppose we have a string fixed at both ends



At $t=0$, it has some arbitrary shape $y = \phi(x)$. :



$\Rightarrow y(x,0) = \phi(x)$.

And at $t=0$, the shape of string is held by hand at rest, then released immediately after.

What is the subsequent motion / shape of the string ($t>0$)?

Sol'n: Initial condition : (At $t=0$) :

$$y(x,0) = \phi(x) = \sum_{n=1}^{\infty} y_n(x,0)$$

$$= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) \quad \ldots \quad (1)$$

And also :
$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0 \quad (\text{for all } x)$$

since all parts of string are initially at rest.

$$\Rightarrow 0 = \sum_{n=1}^{\infty} \left.\frac{\partial y_n}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} \frac{n\pi V}{L} B_n \underbrace{\cos\left(\frac{n\pi V \cdot 0}{L}\right)}_{t=0} \sin\left(\frac{n\pi x}{L}\right).$$

$$\Rightarrow 0 = \sum_{n=1}^{\infty} \underbrace{\left(B_n \frac{n\pi V}{L}\right)}_{\text{constant for each } n.} \sin\left(\frac{n\pi x}{L}\right) \quad \ldots \quad (2)$$

claim: we can use eqns 1 & 2 to figure out what A_n and B_n must be for each n.

To do so, we need to use the following properties of sin & cos:

consider:

where n and m are 2 positive integers. $n, m > 0$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_0^\pi \frac{L}{\pi}\sin(nz)\sin(mz)\,dz$$

← by change of variables:

$z = \frac{\pi x}{L}$

$\Rightarrow dz = \frac{\pi}{L}dx$

$$= \frac{L}{\pi}\int_0^\pi \sin(nz)\sin(mz)\,dz$$

$$= \frac{L}{\pi}\int_0^\pi \left\{\frac{\cos((n-m)z) - \cos((n+m)z)}{2}\right\}dz$$

← ∵ $\cos((n-m)z) = \cos(nz)\cos(mz) + \sin(nz)\sin(mz)$

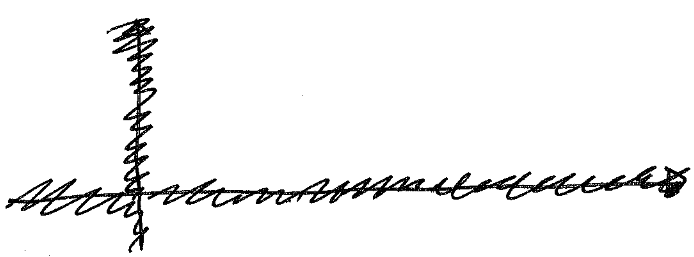
$\cos((n+m)z) = \cos(nz)\cos(mz) - \sin(nz)\sin(mz)$

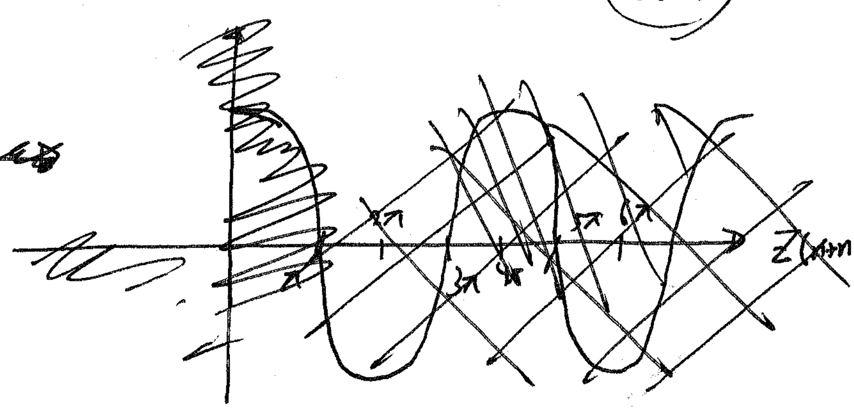
$$= \frac{L}{2\pi}\left\{\int_0^\pi \cos[(n-m)z]\,dz + \int_0^\pi \cos[(n+m)z]\,dz\right\}$$

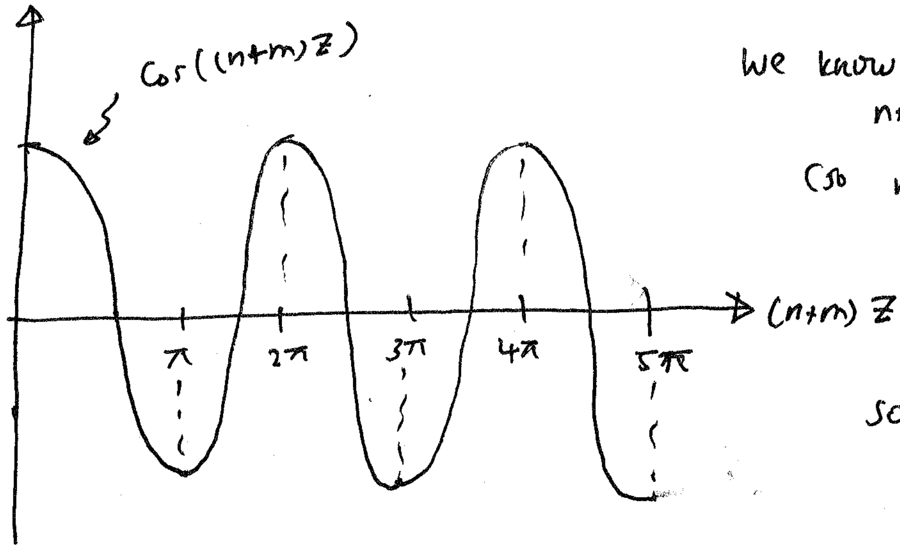
Now, we could just evaluate these mechanically, but we can also see graphically that they must be zero if $n \neq m$.

To see this, first look at graph of $\cos[(n+m)z]$: ~~on inter~~

over







We know that
$n+m \geq 2 \because n \geq 1,\ m \geq 1$
(so $n+m=2$
$n+m=3$,
$\vdots$
so on.)

so, $\int_0^\pi \cos[(n+m)z]\,dz$
$\therefore = 0$.
(It encloses equal areas above and below the horizontal axis: $(n+m)z$.)

So, regardless of what values n & m are,

$$\int_0^\pi \cos[(n+m)z]\,dz = 0.$$

Now, what about $\int_0^\pi \cos[(n-m)z]\,dz$?

If $n \neq m$: Then same story as above: since $|n-m| = 1$ or 2 or 3 $\vdots$

So plotting $\cos[(n-m)z]$ with respect to "x-axis" $= (n-m)z$ would lead you to the same conclusion as above: $\int_0^\pi dz \cos((n-m)z) = 0$ (if $n \neq m$).

If $n=m$: Then $\int_0^\pi \cos[(n-m)z]\,dz = \int_0^\pi dz = \pi$.

$\therefore$ Going back to eqn on pg 101:

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$= \frac{L}{2\pi}\left\{\underbrace{\int_0^{\pi} \cos[(n-m)z]\,dz}_{=\begin{cases}\pi & \text{if } n=m\\ 0 & \text{if } n\neq m\end{cases}} + \underbrace{\int_0^{\pi}\cos[(n+m)z]\,dz}_{0}\right\}$$

$$= \frac{L}{2\pi}\times\begin{cases}\pi, & (n=m)\\ 0, & (n\neq m).\end{cases}$$

$$= \boxed{\begin{cases} L/2, & \text{if } n=m\\ 0, & \text{if } n\neq m.\end{cases}}$$

Why is this useful? Going back to pg 100:

see eqn ①: $y(x,0) = \phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

So:
$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\phi(x)\,dx$$

$$= \sum_{n=1}^{\infty}\left\{\int_0^L \left(\sin\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)A_n\,dx\right\}$$

$$= \sum_{n=1}^{\infty} A_n \left\{\underbrace{\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)}\right\}$$

$\hookrightarrow$ equals zero for $n\neq m$. so all but one term ($n=m$) in this infinite series is non zero.

$$= \frac{A_m L}{2}.$$

so:
$$\boxed{A_n = \frac{2}{L}\int_0^L dx\ \phi(x)\sin\left(\frac{n\pi x}{L}\right)}$$
← This is how you find the A_n's.

Now, looking at eqn ② on pg 100:

$$0 = \sum_{n=1}^{\infty} \underbrace{\left(B_n \frac{n\pi V}{L}\right)}_{\tilde{B}_n} \sin\left(\frac{n\pi x}{L}\right)$$

$\tilde{B}_n$ ← ∵ just a constant

$$\Rightarrow \underbrace{0}_{\text{call } \psi(x)} = \sum_{n=1}^{\infty} \tilde{B}_n \sin\left(\frac{n\pi x}{L}\right)$$

: "Looks" the same as eqn ① (Namely, a Fourier sine series)

Then by same reasoning above:

$$\boxed{\tilde{B}_n = \frac{2}{L}\int_0^L dx\ \psi(x)\sin\left(\frac{n\pi x}{L}\right)}$$

But in this specific case, $\psi(x) = 0$.

$$\Rightarrow \tilde{B}_n = 0$$
$$\Rightarrow \boxed{B_n = 0} \text{ for all } n = 1, 2, \ldots$$

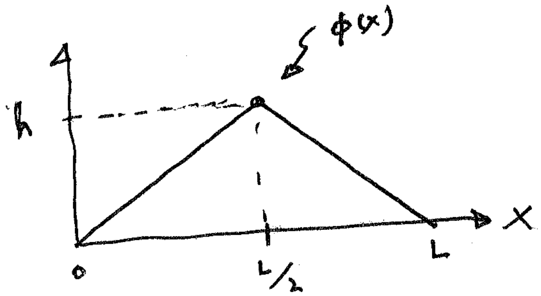
so:
$$\boxed{y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi v t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}$$

where we find A_n using $\boxed{A_n = \frac{2}{L}\int_0^L dx\ \phi(x)\sin\left(\frac{n\pi x}{L}\right)}$.

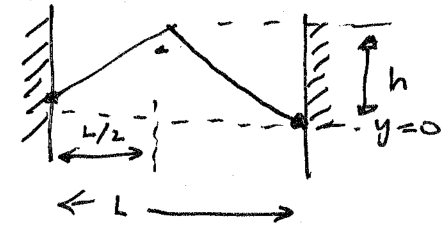
Given an "easy enough" function $\phi(x)$, we can evaluate above integral and find explicit form for A_n.

Ex: Plucked string : plucked in the middle :

$$y(x, t=0) = \phi(x) = \begin{cases} \dfrac{2xh}{L}, & 0 \le x \le L/2 \\[2mm] \dfrac{2h(L-x)}{L}, & \dfrac{L}{2} \le x \le L. \end{cases}$$



$\Rightarrow$



And assume initially held at rest. Then released just after $t=0$.

What's the subsequent shape of the string? $(t>0)$

From pg104 : $\displaystyle y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi v t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$

$$A_n = \frac{2}{L}\int_0^L dx\, \phi(x) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{L}\left[\int_0^{L/2} dx\, \frac{2xh}{L} \sin\left(\frac{n\pi x}{L}\right) + \int_{L/2}^{L} dx\, \frac{2h(L-x)}{L} \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$= \frac{8h}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

$\Rightarrow$ If $n = 2, 4, 6, 8, \ldots$ (~~odd~~ even)

$\Rightarrow \quad A_n = 0 \quad \because \sin(n\pi/2) = 0.$

If $n = 1, 3, 5, \ldots$ (odd)

$\Rightarrow \quad A_n = \dfrac{8h}{(n\pi)^2}(-1)^{n+1}$

$\therefore$

Thus: the general solution is: (i.e. shape of string at subsequent time t > 0) (t > 0)

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi v t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1,3,5,\ldots} A_n \cos\left(\frac{n\pi v t}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \boxed{y(x,t) = \sum_{n=1,3,5,\ldots} \frac{8h(-1)^{n+1}}{(n\pi)^2} \cos\left(\frac{n\pi v t}{L}\right) \sin\left(\frac{n\pi x}{L}\right).}$$

↑ describes shape of string at later time t.