MITES 2008 : Physics III : Oscillations & waves — Hyun Youk

Solution set #3

(pg 1)
(July 16, 08)

Problem 1

Total energy of "slinky" E_{tot}:

1(a)

$$E_{tot} = \underbrace{\sum_{j=1}^{N} \frac{m\dot{y}_j^2}{2}}_{\text{total kinetic energy}} + \underbrace{\sum_{j=1}^{N+1} \frac{k}{2}(y_j - y_{j-1})^2}_{\text{total potential energy}}$$

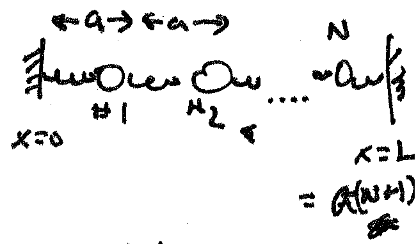
← runs from $j=1$ to $j=N+1$ since there are $N+1$ springs.

Where, $y_0 = y_{N+1} = 0$ ← Boundary conditions (2 walls)

(Notice that there are no particles # 0 and # N+1.) This is same as setting $y_0 = y_{N+1} = 0$ and still have above equation for total energy work out.)

First, the KE_{tot}:

$$\rho = \text{mass density (uniform)} = \frac{M_{total}}{L} = \frac{N \cdot m}{(N+1)a}$$



$$\approx \frac{N \cdot m}{Na} \quad \leftarrow \because N \gg 1 \text{ so } N+1 \approx N$$

$$\rho = \boxed{m/a}$$

$$\therefore KE_{tot} = \sum_{j=1}^{N} \frac{m\dot{y}_j^2}{2} = \sum_{j=1}^{N} \frac{(\rho a)\dot{y}_j^2}{2}$$

$$= \sum_{j=1}^{N} \frac{\rho \dot{y}_j^2}{2}(\delta x) \quad \leftarrow a = \delta x \leftarrow \text{very small (near infinitesimal)}$$

$$= \boxed{\int_0^L \frac{\rho\left(\frac{\partial y}{\partial t}\right)^2}{2} dx}$$

↑ total KE.

← $y_j(t) \to y(x_j, t)$ where $x_j = j \cdot a$

$x_1 = a \approx 0$

$x_N = Na \approx (N+1)a = L$

Next, potential energy total :

$$\sum_{j=1}^{N+1} \frac{k(y_j - y_{j-1})^2}{2}$$

$(y_0 = y_{N+1} = 0) \leftarrow$ Boundary conditions.

$$= \sum_{j=1}^{N+1} \frac{k(\delta y_j)^2}{2}$$

$\leftarrow \delta y_j(t) = y_j(t) - y_{j-1}(t)$

$\uparrow$ small since j^{th} and $(j-1)$-st particles are close to each other (i.e. the separation between their equilibrium positions is $x_j - x_{j-1} = a = \delta x \leftarrow$ small) so we expect a smooth variation in displacement y from one particle to the next.

$$= \sum_{j=1}^{N+1} \frac{ka}{2a} (\delta y_j)^2$$

$$= \sum_{j=1}^{N+1} \frac{ka^2}{2} \left(\frac{\delta y_j}{\delta x}\right)^2$$

$$= \sum_{j=1}^{N+1} \frac{(ka)}{2} \left(\frac{\delta y_j}{dx}\right)^2 \underbrace{dx}_{a = \delta x}$$

$$= \boxed{\int_0^L \frac{(ka)}{2} \left(\frac{\partial y}{\partial x}\right)^2 dx}$$

Express this in terms of things we can measure more easily. (k hard to measure but the force felt by the wall $= ka$ is easier to measure.)

make $y_j \to y(x_j, t)$ (continuum limit)

$\uparrow$ total PE.

$$\therefore \boxed{E_{tot} = \int_0^L \frac{\rho}{2} \left(\frac{\partial y}{\partial t}\right)^2 dx + \int_0^L \frac{(ka)}{2} \left(\frac{\partial y}{\partial x}\right)^2 dx}$$

$$\boxed{\rho = m/a \qquad ka \equiv D}$$

$\uparrow$ in Pset handout notation.

uniform ρ = mass density
$D \equiv$ force felt by a wall due to spring.

$a = \frac{L}{N+1} \approx \frac{L}{N} \quad (N \gg 1)$

N large means that it's large enough so that $a = \frac{L}{N}$ is small.

$\boxed{(b)}$ This was done in class and worked out fully in lecture notes. But just to remind ourselves what we did:

- Start from Newton's 2^{nd} law applied to each (j^{th}) atom:

$$m\ddot{y}_j = -k(y_j - y_{j-1}) + k(y_{j+1} - y_j)$$

$$\Rightarrow \ddot{y}_j = \frac{-k}{m}(y_j - y_{j-1}) + \frac{k}{m}(y_{j+1} - y_j)$$

$$= \frac{k}{m}\left\{(y_{j+1} - y_j) - (y_j - y_{j-1})\right\}$$

$$= \frac{ka}{m}\left\{\frac{(y_{j+1} - y_j)}{a} - \frac{(y_j - y_{j-1})}{a}\right\}$$

$$= \frac{ka}{m}\left\{\frac{y_{j+1} - y_j}{\delta x} - \frac{(y_j - y_{j-1})}{\delta x}\right\} \qquad \begin{array}{l} a \text{ small} \\ a = \delta x. \end{array}$$

But for the limit that $N \to \infty$ (very large ~~#~~ particles), $a = \frac{L}{N+1}$ small, and ~~$\Rightarrow$~~ atoms are very closely packed together. We ~~~~ therefore reasonably ~~~~ expect that there's a smooth variation in displacement of the atoms from one atom to its next neighbor.

$$\Rightarrow y_{j+1} - y_j = \delta y_j \quad \leftarrow \text{small}$$

& Continuum limit

so: $$\ddot{y}_j = \frac{ka}{m}\left\{\frac{\delta y_j}{\delta x} - \frac{\delta y_{j-1}}{\delta x}\right\}$$

$$= \frac{ka}{(m/a)} \cdot \frac{1}{a}\left\{\frac{\delta y_j}{\delta x} - \frac{\delta y_{j-1}}{\delta x}\right\}$$

uniform density of slinky.

over

So: $\ddot{y}_j = \frac{ka}{\rho} \underbrace{\left\{ \frac{\frac{\delta y_j}{\delta x} - \frac{\delta y_{j-1}}{\delta x}}{\delta x} \right\}}_{''}$ $\qquad \rho = m/a$

By definition, 2^{nd} derivative of y with respect to x

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = \frac{ka}{\rho} \frac{\partial^2 y}{\partial x^2}}$$

In this last step, we replaced

$y_j(t) \to y(x_j, t)$

And since $x_{j+1} - x_j = a = \delta x$ ← small,

we can treat x_j as a continuous variable x

(Another way of saying continuum limit)

$$\boxed{ka/\rho = v^2 \leftarrow \text{wave speed.}}$$

$\boxed{\text{c}}$ To solve the wave eqn; I solve the (easier) $\mathbb{C}$-# equivalent

wave eqn: $\frac{\partial^2 \tilde{y}}{\partial t^2} = v^2 \frac{\partial^2 \tilde{y}}{\partial x^2}$ where $\underset{\mathbb{R}}{y} \to \underset{\mathbb{C}}{\tilde{y}}$

Guess: $\tilde{y}(x,t) = \underrightarrow{\tilde{A} e^{i(kx - \omega t)}} + \underleftarrow{\tilde{B} e^{i(-kx - \omega t)}}$ $\qquad v = \sqrt{ka/\rho}$.

$\tilde{A}$, $\tilde{B}$ constants

$\tilde{A} e^{i(kx-\omega t)}$ represents right-moving plane wave

$\tilde{B} e^{i(-kx-\omega t)}$ represents left-moving plane wave.

Notice that without even plugging our guess $\tilde{y}(x,t)$ into the $\mathbb{C}$ - wave equation, we know that it must be a solution to the wave eq'n since it has the form:

$$\tilde{y}(x,t) = \underbrace{f(x-vt)}_{\substack{\uparrow \\ \tilde{A}e^{i(kx-\omega t)} \\ = \tilde{A}e^{ik(x-\frac{\omega}{k}t)} \\ = \tilde{A}e^{ik(x-vt)}}} + \underbrace{g(x+vt)}_{\substack{\uparrow \\ \tilde{B}e^{i(-kx-\omega t)} \\ = \tilde{B}e^{-ik(x+\frac{\omega}{k}t)} \\ = \tilde{B}e^{-ik(x+vt)}}}$$

And we know that any function of the form $f(x-vt)$ and $g(x+vt)$ must each be solution to the wave eq'n (see lecture notes).

Also, the addition (superposition) of the 2 must also be a solution since wave eq'n is linear.

But, what values are allowed for "k", wave number? Can it be any number?

Ans No. Since we have 2 boundary conditions (BCs)

$$y(x=0,t)=0 \qquad y(x=L,t)=0$$

$$\Rightarrow \quad \tilde{y}(x=0,t)=0 \qquad \tilde{y}(x=L,t)=0$$

These 2 BCs constrain what values of k are allowed.

so, BC ① gives: $\tilde{y}(x=0,t)=0$

$$= \tilde{A}e^{-i\omega t} + \tilde{B}e^{-i\omega t}$$

$$= (\tilde{A}+\tilde{B})e^{-i\omega t} \quad \leftarrow \text{should be true for all } t.$$

$$\Rightarrow \quad 0 = \tilde{A}+\tilde{B} \quad \Rightarrow \quad \tilde{A} = -\tilde{B}$$

BC ② gives: $\tilde{y}(x=L,t)=0$

$$= \{\tilde{A}e^{ikL} + \tilde{B}e^{-ikL}\}e^{-i\omega t}$$

$$\Rightarrow \quad 0 = \{\tilde{A}e^{ikL} - \tilde{A}e^{-ikL}\}e^{-i\omega t}$$

& from BC #1

$$= \tilde{A}(e^{ikL} - e^{-ikL})e^{-i\omega t}$$

$$\Rightarrow \quad 0 = 2\tilde{A}i\sin(kL)e^{-i\omega t}$$

∴ & must hold for all t.

$$2\tilde{A}i\sin(kL) = 0$$

But $\tilde{A} \neq 0$ (since $\tilde{A}=0$ means $\tilde{y}(x,t) = \tilde{A}e^{i(kx-\omega t)} + \tilde{B}e^{i(-kx-\omega t)}$

$$= \tilde{A}(e^{i(kx-\omega t)} - e^{i(-kx-\omega t)})$$

$$= 0.$$

trivial, uninteresting solution to wave eqn.)

so,

we need $\boxed{\sin(kL) = 0}$

$$\Rightarrow \quad k_n L = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\Rightarrow \quad k_n = \frac{n\pi}{L}$$

so, each n-value gives a distinct solution $\tilde{y}(x,t)$.

Label these solutions as $\tilde{y}_n(x,t)$.

$$\Rightarrow \quad \tilde{y}_n(x,t) = \tilde{A}_n e^{i(k_n x - \omega_n t)} + \tilde{B}_n e^{i(-k_n x - \omega_n t)} \qquad \Bigg\| \quad k_n = \frac{n\pi}{L}, \quad \omega_n = k_n \cdot v$$

$$= \tilde{A}_n e^{-i\omega_n t} \left\{ e^{ik_n x} - e^{-ik_n x} \right\} \quad \leftarrow \because \tilde{A}_n = -\tilde{B}_n$$

$$= 2i\tilde{A}_n e^{-i\omega_n t} \sin(k_n x)$$

$$= 2i\tilde{A}_n \sin(k_n x) \left\{ \cos(\omega_n t) - i\sin(\omega_n t) \right\}$$

$$= \Big[\underbrace{(2i\tilde{A}_n)}_{``\tilde{B}_n} \cos(\omega_n t) + \underbrace{(2\tilde{A}_n)}_{``\tilde{C}_n} \sin(\omega_n t) \Big] \sin(k_n x) \quad \leftarrow \text{relabel the constants.}$$

This is the $\mathbb{C}$-solution.

What about the real, physically meaningful solution?

well, $\boxed{y_n(x,t) = \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \sin(k_n x)}$

where A_n and B_n are any arbitrary constants (real-valued) is a solution to the real wave eqn.

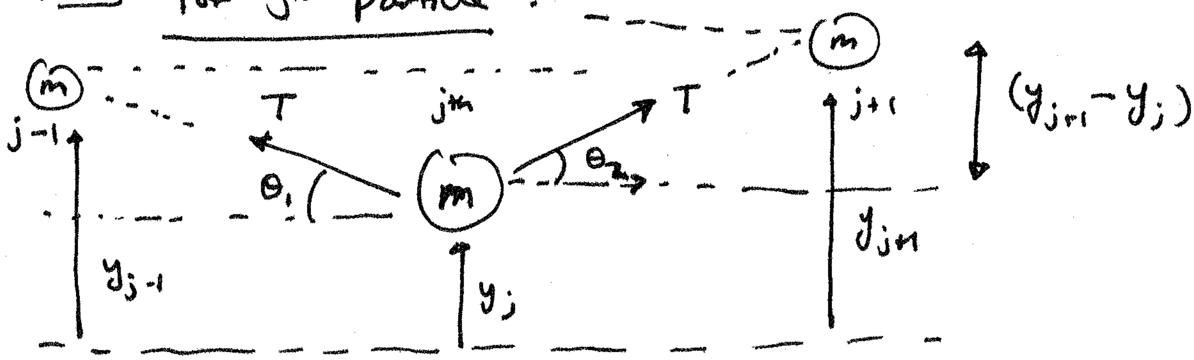
So, the general solution is:

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$$

$$\Rightarrow \boxed{y(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi v t}{L}\right) + B_n \sin\left(\frac{n\pi v t}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)}$$

Problem 2

(a) For j^{th} particle :



First, show that no net horizontal force is on the j^{th} atom :

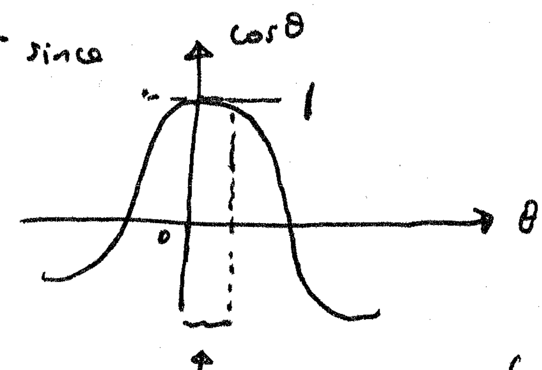
On the j^{th} particle:

$$F_{x\text{-component}} = T\cos\theta_1 - T\cos\theta_2 = T(\cos\theta_1 - \cos\theta_2)$$
$$\approx T(1-1)$$
$$= 0$$

since only small vertical oscillations of each atom are allowed, θ_1 and θ_2 are both very small

$(|\theta_1|, |\theta_2| \ll 1)$

Then $\left.\begin{array}{l}\cos\theta_1 \approx 1 \\ \cos\theta_2 \approx 1\end{array}\right\}$ ← since



↑ In this region $(|\theta| \ll 1)$ $\cos\theta \approx 1$.

∴ No net horizontal component force.

⇒ Only concerned with motion in vertical direction.

Looking at the diagram on pg 8, we can apply Newton's 2nd law to jth particle in vertical direction:

$$m\ddot{y}_j = T\sin\theta_1 + T\sin\theta_2$$

$$= T(\sin\theta_1 + \sin\theta_2)$$

$$\left\{ \begin{array}{l} \sin\theta_1 \approx \dfrac{y_{j-1} - y_j}{a} \\ \sin\theta_2 \approx \dfrac{y_{j+1} - y_j}{a} \end{array} \right.$$

$$= T\left(\frac{y_{j-1} - y_j}{a} + \frac{y_{j+1} - y_j}{a} \right)$$

$$= \frac{T}{a}\left[y_{j-1} - 2y_j + y_{j+1} \right]$$

$$\Rightarrow \boxed{\ddot{y}_j = \frac{T}{ma}\left[y_{j-1} - 2y_j + y_{j+1} \right]} \leftarrow \text{EOM.}$$

Boundary conditions: ① $y_0 = 0$ ② $y_{N+1} = 0$ } (i.e. There isn't a "0th" or "N+1"st particle.)

Thus, above EOM for $j=1$ becomes: (plug in $y_0 = 0$ into y_{j-1} $(j=1)$)

$$\boxed{\ddot{y}_1 = \frac{T}{ma}\left[-2y_1 + y_2 \right]}$$

and for $j = N$: (plug in $y_{N+1} = 0$ into y_{j+1} $(j=N)$):

$$\boxed{\ddot{y}_N = \frac{T}{ma}\left[y_{N-1} - 2y_N \right]}$$

1(b) $\ddot{y}_j = \frac{T}{ma}[y_{j-1} - 2y_j + y_{j+1}]$

$$= \frac{T}{m}\left[\frac{(y_{j+1}-y_j)}{a} - \frac{(y_j - y_{j-1})}{a}\right]$$

Continuum limit: N is so large (take $N \to \infty$ limit) such that "a" is small (near infinitesimal)

$$a = \frac{L}{N+1} \approx \frac{L}{N} \quad (N \gg 1)$$

$$= \delta x \leftarrow \text{small.}$$

i.e.

Then atoms are very tightly, closely packed, and instead of a "rigid" rods making up a necklace, we have a smoothly varying, continuous string. Then, $y_j - y_{j-1} = \delta y_j$

(i.e. displacement of jth atom is not much different from displacement of $j-1$st particle. (Displacement of atoms are smoothly varying from one particle to next.)

This is what we mean by "continuum limit.".

Then we can write $y_j(t) \to y(x_j, t)$

$x_j = ja$.

so $\ddot{y}_j = \frac{T}{m}\left[\frac{\delta y_j}{\delta x} - \frac{\delta y_{j-1}}{\delta x}\right]$

$$= \frac{T}{(m/a)}\frac{1}{a}\left[\frac{\delta y_j}{\delta x} - \frac{\delta y_{j-1}}{\delta x}\right]$$

$$= \frac{T}{\rho}\left(\frac{\left[\frac{\delta y_j}{\delta x} - \frac{\delta y_{j-1}}{\delta x}\right]}{\delta x}\right)$$

$\rho = m/a$

2nd partial derivative with respect to x, by definition

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}} \Rightarrow \boxed{v^2 = T/\rho}$$

$\boxed{(c)}$

$\mathbb{C}$– equivalent EOM for the jth particle is $\quad \left(\underset{\mathbb{R}}{y_j} \to \underset{\mathbb{C}}{\tilde{y}_j} \right)$

$$\boxed{\frac{\partial^2 \tilde{y}_j}{\partial t^2} = \frac{T}{ma}\left(\tilde{y}_{j-1} - 2\tilde{y}_j + \tilde{y}_{j+1} \right)}$$

Guess: $\tilde{y}_j(t) = A_j \exp(i\omega t)$ $\quad A_j =$ constant.

substitute into $\mathbb{C}$– eom above:

$$-\omega^2 A_j \cancel{e^{i\omega t}} = \frac{T}{ma}\left\{ A_{j-1}\cancel{e^{i\omega t}} - 2A_j \cancel{e^{i\omega t}} + A_{j+1}\cancel{e^{i\omega t}} \right\}$$

$\Rightarrow$

$$-\frac{ma\omega^2}{T} A_j = A_{j-1} - 2A_j + A_{j+1}$$

$$\Rightarrow \boxed{-A_{j-1} + \left(2 - \frac{ma\omega^2}{T}\right)A_j - A_{j+1} = 0}$$

Since both ends of spring are fixed at the wall:

$$\boxed{A_0 = 0 \qquad A_{N+1} = 0 \qquad (\because y_0 = 0 \quad y_{N+1} = 0)}$$

$\boxed{(d)}$ Rearranging above eq'n:

$$\frac{A_{j-1} + A_{j+1}}{A_j} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}$$

where $\omega_0^2 = \frac{T}{ma}$

Guess: $A_j = Ce^{ij\theta}$ $\qquad C, \theta$ constants.

plugging into above eq'n:

$$\frac{e^{i(j-1)\theta} + e^{i(j+1)\theta}}{e^{ij\theta}} = \frac{2\omega_0^2 - \omega^2}{\omega_0^2}$$

→ over

$$\frac{\cancel{e^{ij\theta}}\left\{e^{-i\theta}+e^{i\theta}\right\}}{\cancel{e^{ij\theta}}} = \frac{2\omega_0^2-\omega^2}{\omega_0^2}$$

$$\boxed{2\cos\theta = \frac{2\omega_0^2-\omega^2}{\omega_0^2}}$$

independent of j.

Boundary conditions: $y_0 = 0$ (BC1) $\quad y_{N+1} = 0$ (BC2)

$\therefore$ BC1: $A_0 = C\sin(j\theta) \quad (j=0)$
$= 0$ ← automatically satisfied.

BC2: $A_{N+1} = C\sin((N+1)\theta)$
$= 0 \quad \Longrightarrow (N+1)\theta_n = n\pi \quad n = 1, 2, \ldots, N.$

$$\Rightarrow \boxed{\theta_n = \frac{n\pi}{N+1}}$$

Notice that $\cos(j\theta)$ cannot satisfy BC1: since $C\cos(j\theta)$ (when $j=0$)
$= C\cos(0)$
$= C.$
$\neq 0$

(And we can't have $C=0$ since then we get uninteresting solution) ($y \equiv 0$)

Thus,
$$\boxed{\begin{aligned} A_j &= C\sin(j\theta_n) \\ &= C\sin\left(\frac{jn\pi}{N+1}\right)\end{aligned}}$$

To get ω_n :

$$\frac{2\omega_0^2 - \omega_n^2}{\omega_0^2} = 2\cos\theta_n = 2\cos\left(\frac{n\pi}{N+1}\right)$$

$$\Rightarrow \boxed{\omega_n^2 = 2\omega_0^2\left[1-\cos\left(\frac{n\pi}{N+1}\right)\right] \quad (n=1,2,3,\ldots N)}$$

Notice that: $A_j = C\sin\left(\frac{nj\pi}{N+1}\right)$ is the amplitude of the jth mass at fixed normal mode n.

In the continuum limit, these become the standing waves.