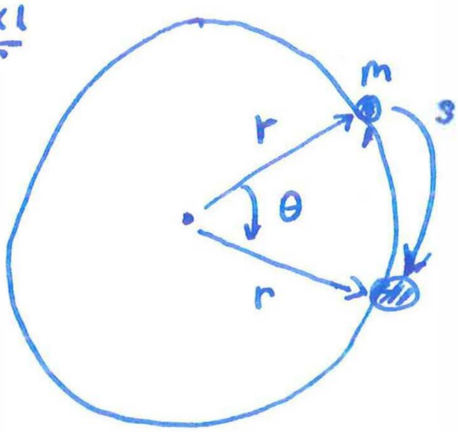
Lecture - Rotational Motion

20-Dec-2016
[Tues]

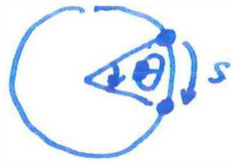
Ex 1



circle.

$r\theta = s$

Arc length



In a circle, r = constant over time.

$\theta = \theta(t)$ ← changes over time

$s = s(t)$ ← changes over time.

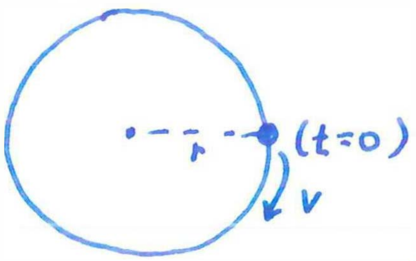
So $r\,\theta(t) = s(t)$

$\Rightarrow \theta(t) = \dfrac{s(t)}{r}$

θ measured in radians. (rad)

0 rad.	π rad	2π rad
$=$	$=$	$=$
$0°$	$180°$.	$360°$

Ex



Say v = constant over time (linear speed)

Say particle goes around clockwise at this constant speed.

~~Then~~

Then $vT = 2\pi r$ $\quad T$ = time taken to go 1 full circle.

$\Rightarrow T = \dfrac{2\pi r}{v}$

And: $\theta(t) = \dfrac{s(t)}{r} \Rightarrow \theta(T) = \dfrac{2\pi r}{r}$

$= 2\pi$

$\theta(2T) = 4\pi$

$= 2\cdot\dfrac{2\pi r}{r}$ $\quad$ and so on ...

Note that in

$vt = s(t)$

so: $\boxed{\theta(t) = \dfrac{vt}{r}}$ ← Angular ~~displacement~~ distance

Angular speed ω ← Greek letter "omega"

$$\frac{\theta(t_2)-\theta(t_1)}{t_2-t_1} = \frac{v(t_2-t_1)}{r(t_2-t_1)}$$

← we can use this (t_2-t_1) because particle moves with constant speed around the circle.

$$\Rightarrow \boxed{\omega = v/r}$$

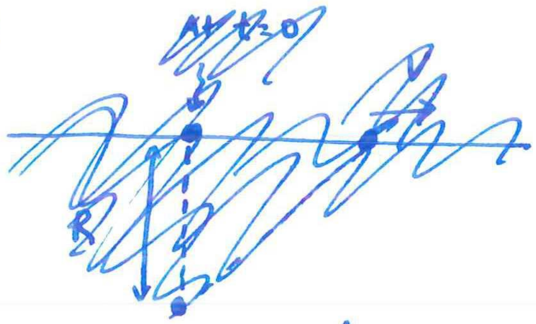
We can also see this by: $\omega = \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{vt}{r}\right)$

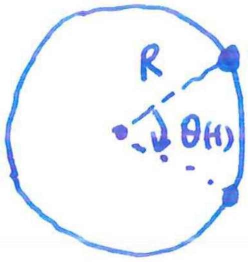
$= \frac{v}{r}$. ← same result as above.

Above analysis was for the particle moving in circle with a constant speed.

☑

Ex 2:



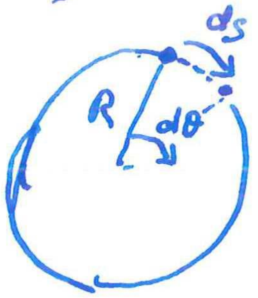


$V(t)$ ← Now no longer constant.

i.e.g. $V(t) = At$ (A=constant)

linearly increasing with time. (i.e. rotating faster as t increases.)

Then:



$V(t)\,dt = ds$

↑ Arc length travelled in small (infinitesimal) time dt.

$V(t)dt = ds$; $ds = R\,d\theta$

So $V(t)\,dt = R\,d\theta$

$$\Rightarrow \boxed{\frac{V(t)}{R} = \frac{d\theta}{dt} = \omega}$$

← Angular speed.

← Result for particle moving in perfect circle at arbitrary (Not necessarily constant) speed.

☑

Ex 3 : Computing Angular acceleration α

In the previous example: $\omega = \frac{d\theta}{dt} = \frac{V(t)}{R}$

so:

$$\frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{V(t)}{R}\right) = \frac{1}{R}\underbrace{\frac{dV}{dt}}_{= a_T} = \frac{1}{R} a_T$$

a_T: "Tangential acceleration."

Here,

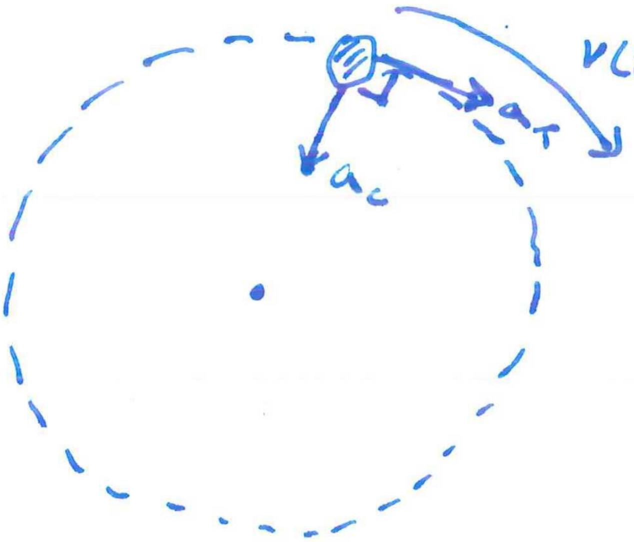
$$\boxed{\alpha \equiv \frac{d\omega}{dt}}$$

"Angular acceleration"

(Greek letter "alpha")

$\boxtimes$

Note that for particle moving in a circle:



there are 2 accelerations

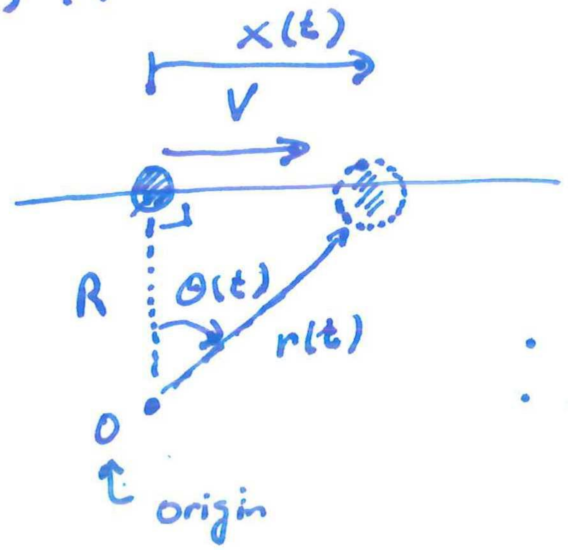
1) a_c = Centripetal acceleration

2) a_T = Tangential acceleration.

What about particles <u>not</u> moving in circles??

<u>Ex 4</u>

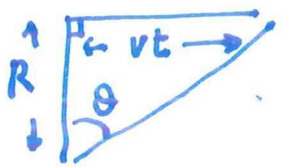
FF



V = Constant speed.

- $r(0) = R$.
- what is angular speed ω?
- what is the angular distance $\theta(t) = ?$

<u>sol'n</u>: $x(t) = vt$



$$\tan\theta = \frac{vt}{R}$$

$$\Rightarrow \boxed{\theta(t) = \text{Arctan}\left(\frac{vt}{R}\right)} \quad \left[\text{Note: } \text{Arctan} = \tan^{-1}\right]$$

↑ Angular distance

$$\omega = \frac{d\theta}{dt} = \frac{d\,\text{Arctan}(u)}{du} \cdot \frac{du}{dt} \qquad u = vt/R.$$

$$= \frac{1}{1+u^2} \cdot \frac{v}{R} \qquad \leftarrow \text{use: } \frac{d\,\text{arctan}(x)}{dx} = \frac{1}{1+x^2}$$

$$\Rightarrow \boxed{\omega = \frac{1}{1+\left(\frac{vt}{R}\right)^2} \cdot \frac{v}{R}}$$

<u>Qu</u>: What happens after a very long time (i.e. $t \to \infty$)? to θ & ω?

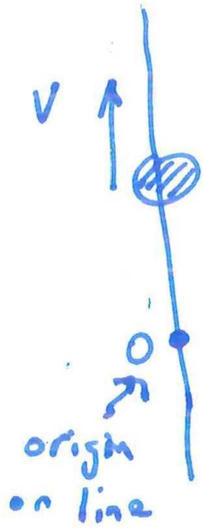
<u>Sol'n</u>:

$$\lim_{t\to\infty} \theta(t) = \text{Arctan}(\infty) = \pi/2.$$

$\lim_{t\to\infty} \omega(t)$: For t very large,

$$\omega(t) \approx \frac{v/R}{(vt/R)^2} = \frac{1}{\frac{v}{R} t^2} \to 0.$$

Ex 5



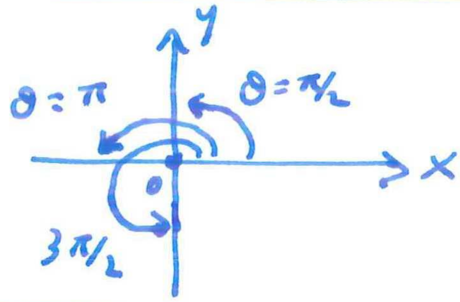
V = constant speed.

Qu: what is $\theta(t)$ and $\omega(t)$?

Soln:

$\theta = \begin{cases} \pi/2 \\ 0 \end{cases}$ ← either one is ok. Depends on how you define your coordinate system.

Polar coordinate system:



Since particle always stays on the line, $\theta(t) = \pi/2$ (or 0). at all times t.

And so $\boxed{\omega = \frac{d\theta}{dt} = 0.}$ (at all times.)

Note that this is true even when V is not constant.

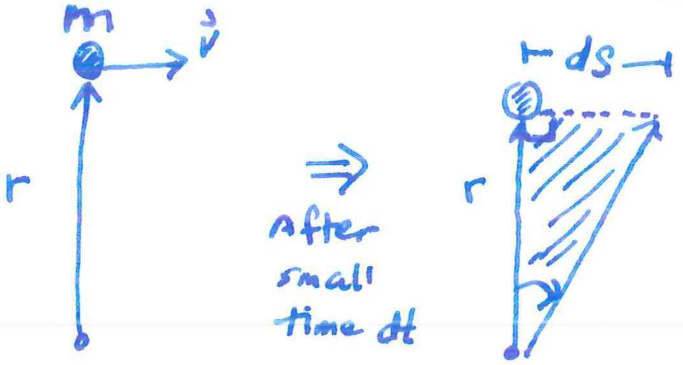
Where do torque & moment of inertia come from?

The book shows one way of motivating where they originate from. Here let's look at them differently.

Recall Lecture 5 (see Lecture notes 5: Pg. 5-28 & 5-29) how we defined linear momentum.

(~~"Quantity of linear motion~~
→ "Quantity of linear motion".
($\vec{p} = m\vec{v}$))

Angular momentum $\equiv$ "Quantity of angular motion"



$ds = v\,dt$

Area of shaded region = Area of triangle

$= \frac{1}{2} r \cdot ds$

$= \frac{1}{2} r \cdot v\,dt.$

So we say per unit time, the particle has the ability to span (sweep out) $\frac{r \cdot v}{2}$ area.

(~~we define~~ On Pg 5-28 & 5-29 of Lecture note 5, we considered particle sweeping/painting linear distance to define linear momentum.)

Like we did with linear momentum, if 1 kg mass sweeps a triangular area of $1\,m^2$ and 2kg mass sweeps the same area per unit time, we want to define angular momentum such that the 2kg particle has twice the angular momentum than the 1kg particle.

For this reason, we multiply the area by the mass m.

so:

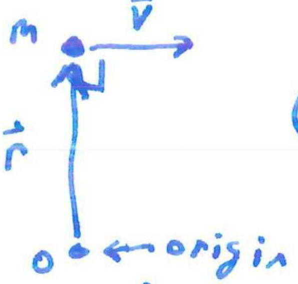
$$m \cdot \left(\text{Area of swept region} \right) / \text{time} = \frac{mrV}{2}.$$

' One more modification to our definition: Let's ~~drop~~ drop the 2 because it's just a number that's constant and it's annoying to keep writing it.

So we get:

$$\boxed{\underset{\vec{L}\ \text{"defined as"}}{mrV.} \equiv \underline{\text{Angular momentum.}}}$$

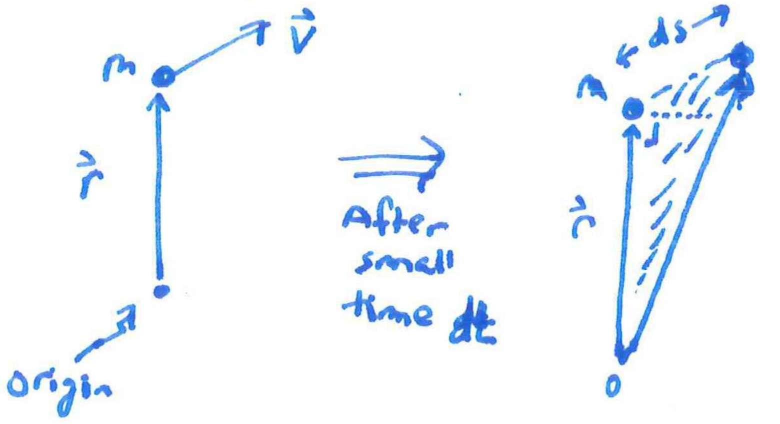
~~Bec this~~ This is for



$(\vec{r} \perp \vec{v})$.

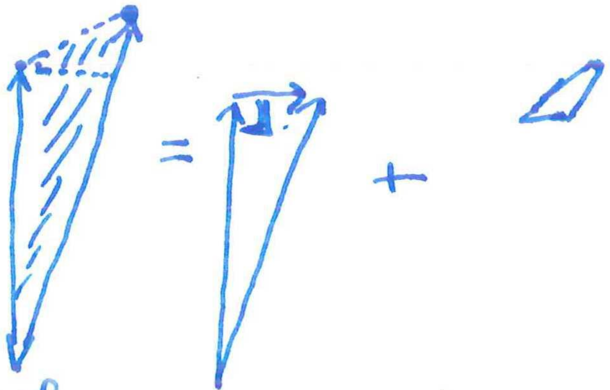
What happens if $\vec{r}$ and $\vec{v}$ are no longer perpendicular to each other?

e.g.



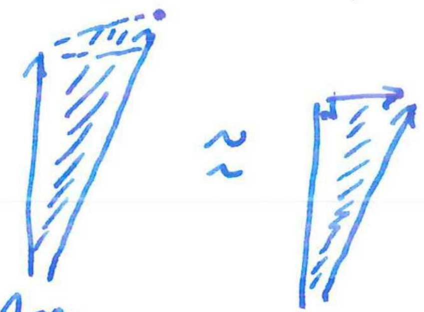
← sweeps shaded area in a more 'complex (not right angle) triangle.

But since dt is very small (infinitesimal):
we have:



Area of triangle = Area of right-angled triangle + Area of tiny triangle.

Idea (can be proved mathematically - Involves Taylor series)
If dt is very tiny (infinitesimal), then basically

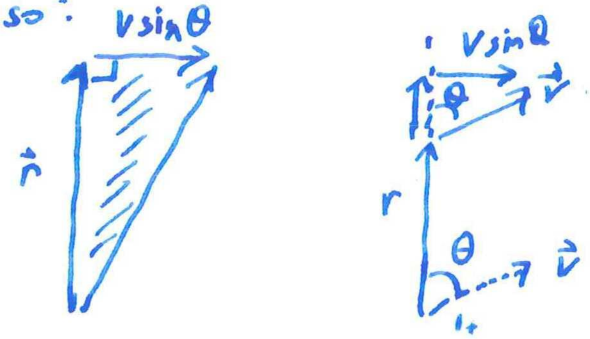


Area $\approx$ Area of right-angled triangle.

like in homework #4.

(Area of the tiny triangle $\to 0$ as $dt \to 0$)

And so:



so: ~~for same argument as in~~.
using the same argument as on Pg 7,
we have: $\boxed{\text{Angular momentum} = mrv\sin\theta}$

* Note: When $\theta = \pi/2$, we obtain mrv result on Pg 7. Makes sense!

Now we can combine results from Pg 7 and Pg 8 to get a more general result:

Note that

$$r v \sin\theta = \underbrace{|\vec{r} \times \vec{v}|}$$

↑ taking the magnitude of the cross-product between $\vec{r}$ and $\vec{v}$ (which is also a vector.)

~~The fact~~ So:

$$m r v \sin\theta = m|\vec{r} \times \vec{v}|.$$

But like in the case of linear momentum, we make Angular momentum to be a vector (has a direction)

Direction represents direction of rotation, defined by $\vec{r} \times \vec{v}$.

So:

$$\boxed{\begin{aligned} \vec{L} &= m\,\vec{r} \times \vec{v} \\ &= \vec{r} \times (m\vec{v}) \\ &= \vec{r} \times \vec{p} \end{aligned}}$$

← True definition of Angular momentum $\vec{L}$.

$\vec{p}$ = linear momentum of particle.

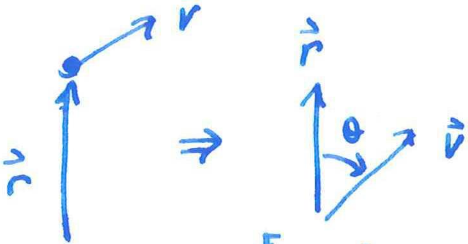
<u>Direction of $\vec{L}$</u>:

Defined by $\vec{r} \times \vec{p}$

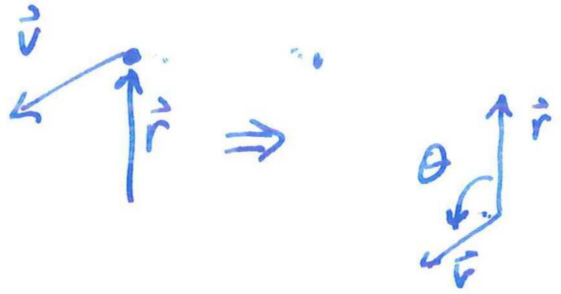
$= \vec{r} \times m\vec{v}$

$= m(\vec{r} \times \vec{v})$ ← use "Right hand rule"

(Defined in chapter 11 of book.)



[curl your right hand, from $\vec{r}$ towards $\vec{v}$. stretch your thumb. It points into the page ⇒ $\vec{L}$ points into the page]



$\vec{r} \times \vec{v} \Rightarrow$ vector points out of the page

(perpendicular to the plane of this paper.)

Torque: Refined as rate of change of Angular Momentum

(~~Rest data~~
Analogy: $\vec{F}$ defined as rate of Change of $\vec{p}$ in Lecture 5 (see Lecture note Pg 5-29))

Define Torque:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$= \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{= 0} + \vec{r} \times \underbrace{\frac{d\vec{p}}{dt}}_{= \vec{F}}$$

~~chain rule~~ product rule

because

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0.$$

$\because \vec{v} \times \vec{v} = |v||v| \sin 0 = 0$

$$= \vec{r} \times \vec{F}.$$

So Torque: $\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$

Summarizing, we derived

$$\boxed{\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ &= m\vec{r} \times \vec{v} \quad \leftarrow \text{Angular momentum.} \\ \vec{\tau} &= \vec{r} \times \vec{F} \quad \leftarrow \text{Torque.} \end{aligned}}$$

Special case: $\vec{v}$ and $\vec{r}$ are perpendicular to each other.

$$\text{so: } |\vec{r} \times \vec{v}| = |\vec{r}||\vec{v}| \sin \pi/2$$

$$= |\vec{r}||\vec{v}|.$$

$$= |\vec{r}||r\omega| \leftarrow \text{from (Pg 3): } V = r\omega$$

$$= r^2 \omega.$$

$$\text{so: } \boxed{|\vec{L}| = \underbrace{(mr^2)}_{\text{call it } I}\omega.}$$

($I \equiv$ moment of inertia)

(also called "rotational inertia")

("Angular" mass)

because

$I\omega$ looks similar to mv

[$I\omega$ (rotational)

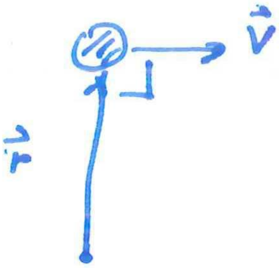
$\updownarrow \; \updownarrow$

mv (linear motion)]

Likewise:

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

But for: $\vec{v} \perp \vec{r}$



we have:

$$\vec{\tau} = \vec{r} \times m\vec{a}_T$$

$$|\vec{\tau}| = r \cdot m r \alpha$$

because $r\omega = v$

So $\alpha = \frac{d\omega}{dt}$

$= \frac{1}{r}\frac{dv}{dt}$

$= \frac{1}{r} a_T$

$$= m r^2 \alpha.$$

$$= \boxed{I \alpha.}$$

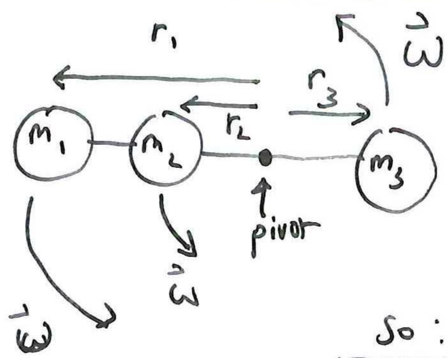
for the special case of $\vec{v} \perp \vec{r}$.

Here $I\alpha$ (rotational)

$\updownarrow$

$m\,a$ (linear motion)

Note:



About the pivot, all masses connected by massless rod rotate together with the same angular frequency $\vec{\omega}$.

So:

$$\boxed{\vec{L}_{total} = m_1 r_1^2 \vec{\omega} + m_2 r_2^2 \vec{\omega} + m_3 r_3^2 \vec{\omega}}$$

$$= \underbrace{[m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2]}_{\text{" } I_{total}} \vec{\omega}$$

total rotational inertia of system.

Ex 6 :

pivot point

massless rod connecting 2 masses.

Qu:

Calculate the rotational inertia (moment of inertia) of above object.

Sol'n

$$\boxed{I_{total} = m_1 r_1^2 + m_2 (L - r_1)^2}$$

Qu:

What's the angular momentum if the system is spinning with angular velocity $\vec{\omega}$ counterclockwise? [Indicate magnitude + direction of $\vec{L}$].

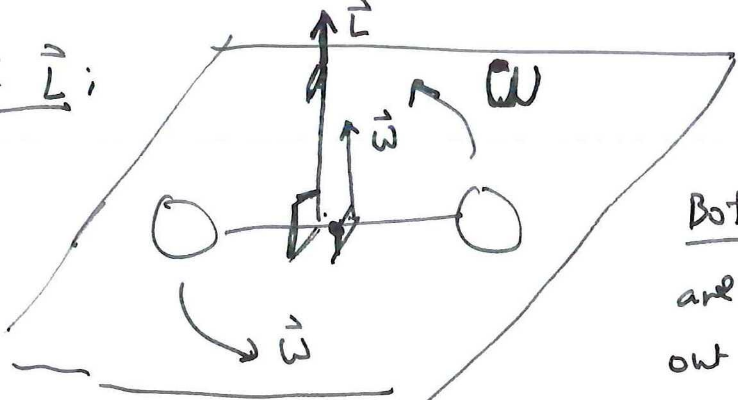
Sol'n:

$$\vec{L} = I_{total} \vec{\omega}$$

$$= [m_1 r_1^2 + m_2 (L - r_1)^2] \vec{\omega}.$$

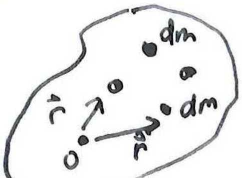
Magnitude is: $|\vec{L}| = [m_1 r_1^2 + m_2 (L - r_1)^2] \omega$. $\omega = |\vec{\omega}|$

Direction of $\vec{L}$:



Both $\vec{\omega}$ and $\vec{L}$ are vectors pointing out of page, perpendicular to the page as drawn here.

For continuous mass distribution



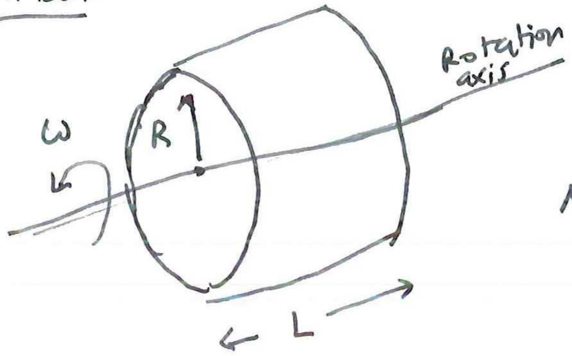
$$I_{total} = \sum_{i=1}^{N=\infty} (dm) \cdot r_i^2 = \int dm \, r^2$$

r_i = distance of i^{th} little mass piece (w/ mass dm) from the pivot.

Ex 7. <u>Rotational inertia of a continuous object</u> :

(Note: you can see rotational inertia for various continuous objects on pg 194 of your book)

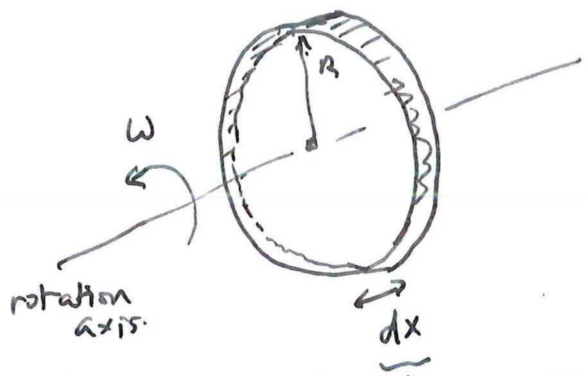
<u>Hollow Cylinder</u> = cylinder with a shell and its inside taken out.



(e.g. toilet roll after all the toilet tissue is gone.)

M = total mass.
uniformly distributed over the cylinder.
(i.e. Constant mass density)

<u>To calculate</u> I ; take a small (infinitesimal) slice :



Let λ = mass density
$= M / \text{surface Area of cylinder}$

$$I_{slice} = \int R^2 \, dm = R^2 \int dm$$

$$= R^2 m \quad \leftarrow \text{mass}$$

$$= R^2 \frac{M}{L} \, dx$$

small thickness of this slice of cylinder.

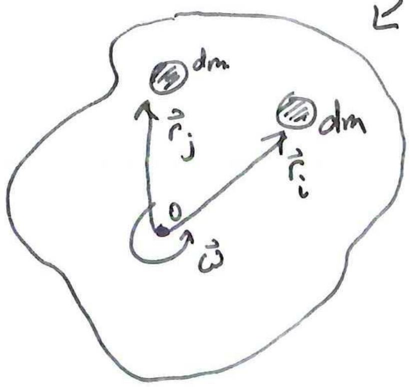
$$I_{total} = \sum I_{slice}$$

$$= \int_{x=0}^{L} \frac{M R^2}{L} \, dx$$

$$= M R^2$$

$$\boxed{I_{total} = MR^2}$$

Ex 8 Rotational kinetic energy.



continuous body

Total mass $M = \int_{\text{volume of object}} dm$

Total kinetic energy:

$$KE_{total} = \sum (\text{KE of each "dm" piece})$$

$$= \int \frac{1}{2}(dm)\, v^2 \qquad v = \text{speed of each dm piece.}$$

$$= \int \frac{1}{2}(dm)\, r^2 \omega^2 \qquad = r_i \omega.$$

$$= \frac{\omega^2}{2} \int (dm)\, r^2$$

Integral done over the ~~volume of~~ entire body of object.

$$= \frac{\omega^2}{2} I_{total}$$

Every dm piece has the same angular velocity $\vec{\omega}$.

(because all ~~com~~ connected to each other.)

$$I_{total} = \int (dm)\, r^2.$$

$$\boxed{KE_{rot} = \frac{I_{total}\, \omega^2}{2}}$$

rotational kinetic energy