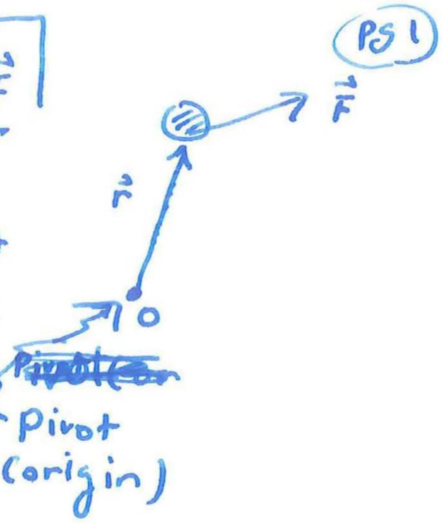
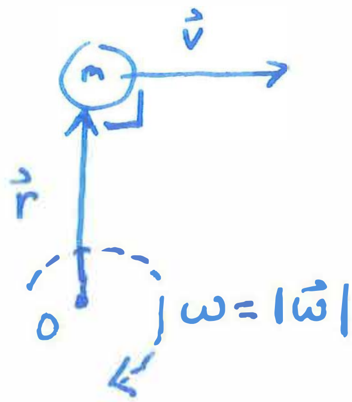
Yesterday, we derived: $\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$

τ Torque (use right hand rule to get direction)



and defined Angular momentum: $\vec{L} = \vec{r} \times \vec{p}$

We then saw that [redacted]
we have:
(we found yesterday): $\boxed{\vec{L} = mr^2 \vec{\omega}}$



$\boxed{I \equiv mr^2}$

I rotational inertia
(also called "moment of inertia")

$\vec{\omega}$ = Angular velocity vector points out of the page
[If m rotates counter clockwise, then $\vec{\omega}$ points into the page.]

Also, we found that:
"I" is like the mass in rotational motion.
By this, we mean that:

Rotational motion: $\vec{L} = I \vec{\omega}$ And $\vec{\tau} = I \vec{\alpha}$

$\updownarrow$ $\updownarrow$ $\updownarrow$

Linear motion: $\vec{p} = m \vec{v}$ $\vec{F} = m \vec{a}$

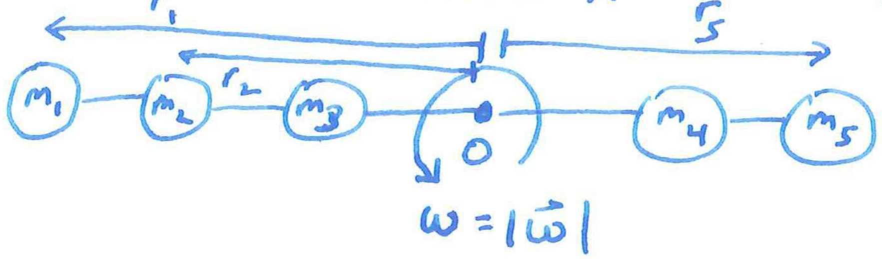
$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$

α Angular acceleration.

Ex: objects joined together + rotating together

Qu: what is the combined object's angular momentum? & ~~or~~ rotational inertia?

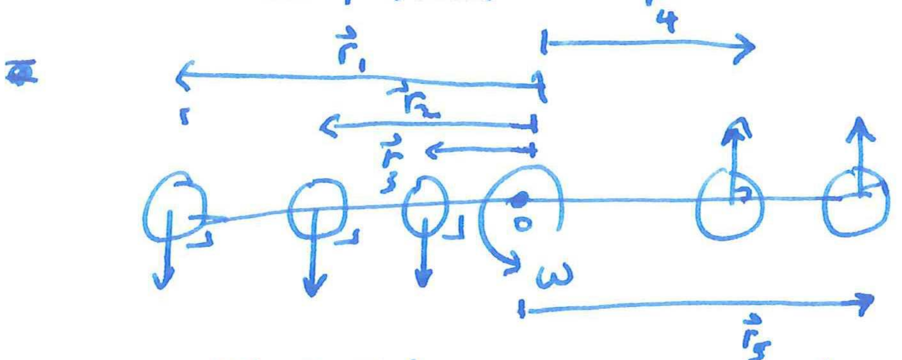
(Also see examples on pgs 12-13 of yesterday's lecture notes).



- Masses joined by rigid massless stick.
- All rotate together with constant angular velocity $\vec{\omega}$.

$$\vec{L}_{total} = \vec{L}_1 + \vec{L}_2 + \dots + \vec{L}_5$$

Angular momentum of each particle



All particles move perpendicular to their position vector ($\vec{r}_i$). All have the same angular velocity $\vec{\omega}$.

So: $\vec{L}_i = m_i r_i^2 \vec{\omega}$

So:

$$\vec{L}_{total} = m_1 r_1^2 \vec{\omega} + m_2 r_2^2 \vec{\omega} + \dots + m_5 r_5^2 \vec{\omega}$$

$$= \underbrace{[m_1 r_1^2 + \dots + m_5 r_5^2]}_{= I_{total}} \vec{\omega}$$

$$I_{total} = \underset{= m_1 r_1^2}{I_1} + I_2 + \dots + \underset{= m_1 r_5^2}{I_5}$$

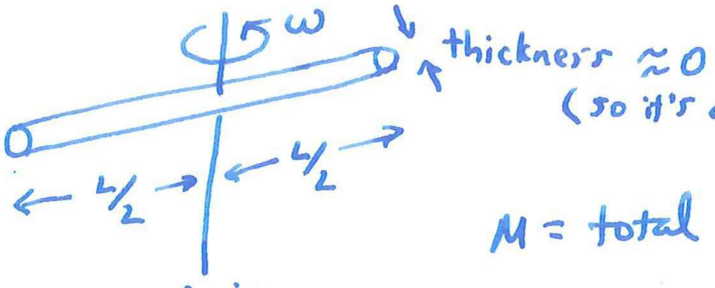
- total rotational inertia I_{total}

$\boxtimes$

Ex 1: Rotational inertia of a continuous object

Thin rod:

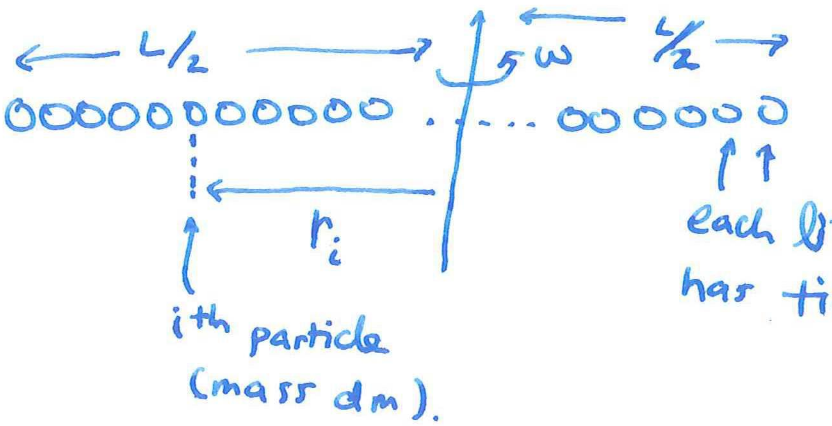
(uniform mass density)



M = total mass of rod.

Qu: Rotational Inertia of thin rod rotating about the axis of rotation as shown?

Sol'n: This is like the previous example but now all the particles are next to each other:



So like the previous example, since all the particles are rotating with the same angular velocity $\vec{\omega}$; we have:

$$I_{total} = I_1 + I_2 + \dots + I_N \quad \text{(N very large)}$$

(N particles make up the rod.)

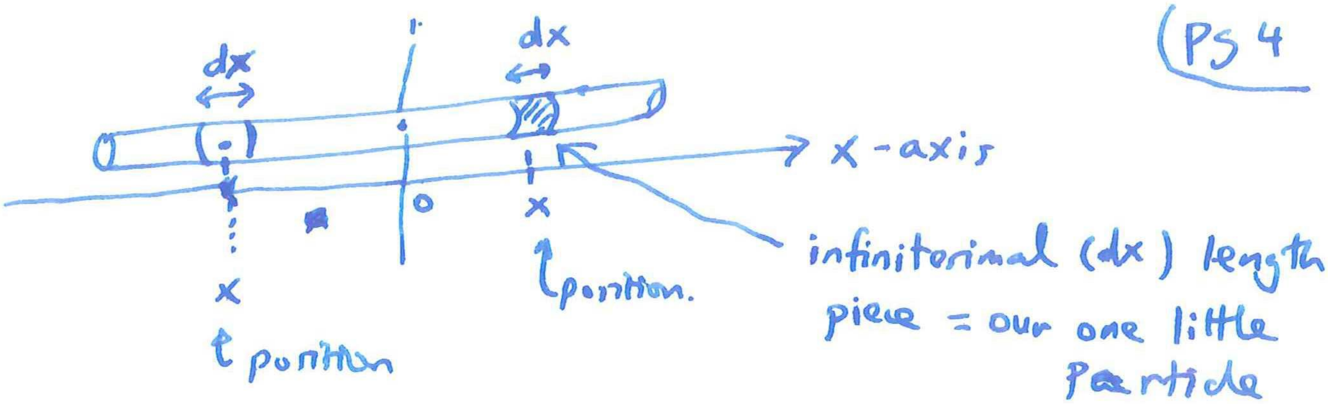
$$= (dm)r_1^2 + (dm)r_2^2 + \dots + (dm)r_N^2$$

$$= \sum_{i=1}^{N} (dm) r_i^2$$

Let $\lambda = M/L$

↑ Mass density (1-dimensional)

(PS 4



so: $dm = \lambda \cdot dx$ (λ = mass/length)

so: $r_i = x$

$$I_{total} = \sum_{i=1}^{N} (\lambda dx) \cdot x^2$$

$$= \lambda \sum_{i=1}^{N} x^2 \cdot dx$$

$\sum_{dx} \rightarrow \int dx$

[whenever you have a sum of infinitesimal quantities, it becomes an integral ← that's in fact the definition of integral.]

$$= \lambda \int_{-L/2}^{L/2} x^2 dx$$

$$= \frac{\lambda x^3}{3} \Big|_{-L/2}^{L/2}$$

$$= \frac{\lambda}{3} \left[\frac{L^3}{8} - \left(-\frac{L}{2}\right)^3 \right]$$

$$= \frac{\lambda}{3} \left[\frac{L^3}{4} \right]$$

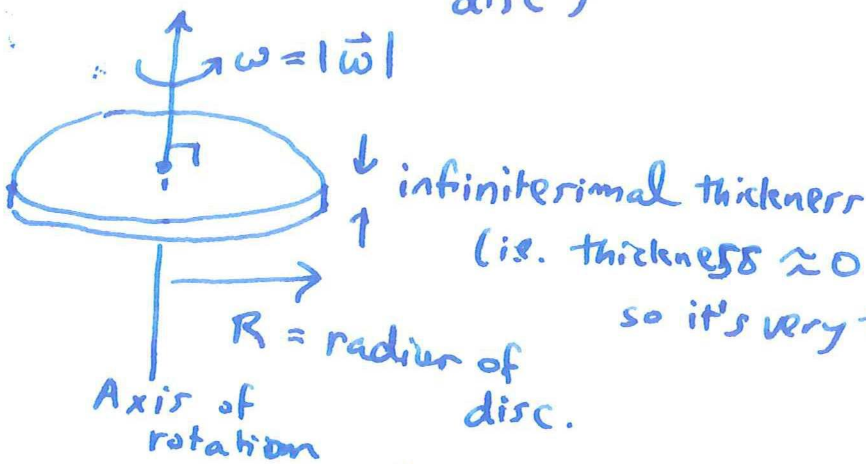
$$= \frac{\lambda L^3}{12}$$

$$= \boxed{\frac{M L^2}{12}}$$

☑

Ex 2: Rotating disc (2-dimensional disc)

M = total mass uniformly distributed over disc

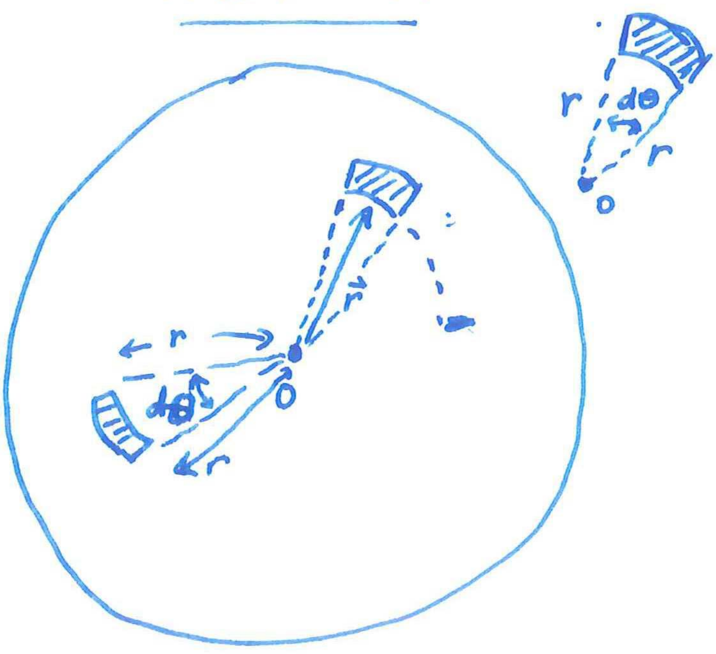


Qu: What is the rotational inertia of this disc with the axis of rotation as shown?

Sol'n We have a 2D problem because thickness ≈ 0

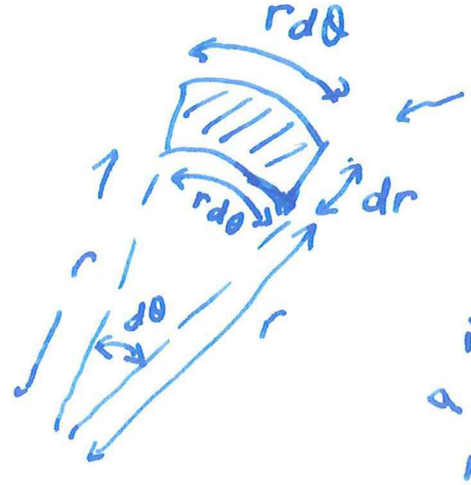
Again, like the previous example, we first think of the disc to be made of many tiny pieces, each with mass dm.

Bird's eye view:
(view from top)



Area of tiny piece $= dA$

$$dA = r(d\theta)(dr)$$



tiny curved rectangle. Since tiny curvature, it's basically a straight rectangle.

So $I_{total} = \sum_{i=1}^{N} I_{\substack{each \\ piece}}$ (N very large)

N pieces make up the disc.

$$= \sum_{i=1}^{N} (dm)\, r_i^2$$

$$\lambda = \frac{M}{\text{Area of disc}} = \frac{M}{\pi R^2}$$

$$= \sum_{i=1}^{N} (\lambda \cdot dA)\, r_i^2$$

$$= \sum_{i=1}^{N} (\lambda \cdot r_i d\theta \cdot dr)\, r_i^2$$

$\sum d\theta\, dr \to \iint dr\, d\theta$ (By definition of integral)

$$= \int_{r=0}^{R}\int_{\theta=0}^{2\pi} \lambda r^3\, d\theta\, dr$$

$$= \int_0^R \lambda\, 2\pi r^3\, dr$$

$$= \cancel{2}\pi\lambda \frac{r^4}{\cancel{4}_{\,2}}\Big|_0^R$$

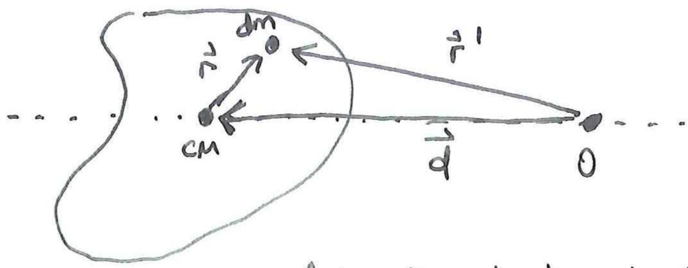
$$= \frac{\pi\lambda}{2} R^4$$

$$= \frac{\cancel{\pi}}{2} \frac{M}{\cancel{\pi} \cancel{R^2}} R^{\cancel{4}\,2}$$

$$= \boxed{\frac{MR^2}{2}}$$

Parallel Axis Theorem.

M = total mass of object.



- CM = center of mass
- O = New origin.
- 2 axes of rotations that are parallel to each other [one goes through CM the other goes through O.].

rotational inertia about axis that goes through point O.

$$I_0 = \int (dm)|\vec{r}\,'|^2$$

summing up all the little masses (each has mass dm.)

$$= \int (dm)\,|\vec{d}+\vec{r}|^2$$

$$= \int dm\,(\vec{d}+\vec{r})\cdot(\vec{d}+\vec{r})$$

definition of dot product.

$$= \int dm\,[\vec{d}\cdot\vec{d} + \vec{d}\cdot\vec{r} + \vec{r}\cdot\vec{d} + \vec{r}\cdot\vec{r}]$$

$$= \int dm\,|\vec{d}|^2 + 2\int dm\,\vec{d}\cdot\vec{r} + \int dm\,|\vec{r}|^2$$

$$= |\vec{d}|^2 \underbrace{\int dm}_{= M} + 2\vec{d}\cdot\left(\int dm\,\vec{r}\right) + \underbrace{\int dm\,|\vec{r}|^2}_{= I_{cm}}$$

M: total mass of object

Take outside because $\vec{d}$ is constant vector.

I_{cm}: rotational inertia about CM.

$$= M|\vec{d}|^2 + I_{cm} + 2\vec{d}\cdot\underbrace{\left(\int dm\,\vec{r}\right)}_{= 0}$$

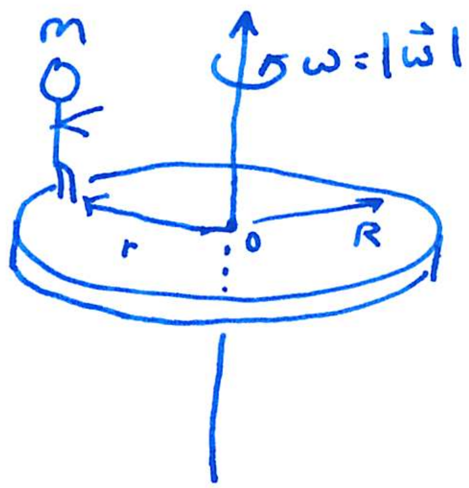
$0 \leftarrow \because \dfrac{\int dm\,\vec{r}}{M}$ = position of cm $= 0$ by definition.

$$\Rightarrow I_0 = \boxed{I_{cm} + Md^2}$$

formula for parallel axis theorem.

Typical problems that involve torque + conservation of $\vec{L}$:

EX 1



- R = radius of rotating disc
- m = mass of person.
- r = radial distance of the person from the axis of rotation.
- I = rotational inertia of disc

- Initially, the person is at rest, at r_1, rotating with the disc.

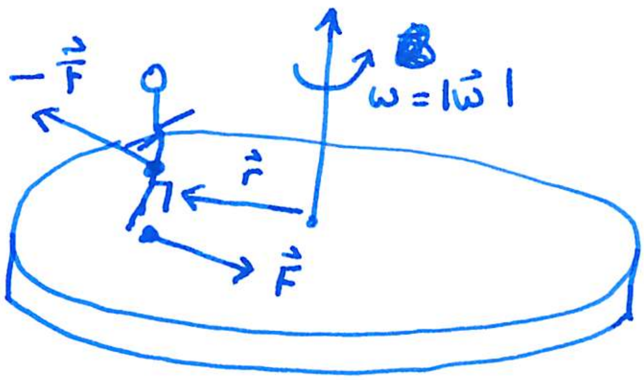
so: $L_{\substack{total\\initial}} = I\omega_i + mr^2\omega_i$

total Ang. momentum at $t=0$

$= [I + mr^2]\,\omega_i$ ω_i = initial angular velocity

- Then, ~~the~~ person walks along an arbitrary path on the rotating wheel.
(e.g. walks towards the axis of rotation.)
While walking, the person exerts force $\vec{F}$ on the ground ~~of~~ (rotating disc). This happens because the person's foot pushes on the ground (to propel himself/herself).
By Newton's 3rd law, the ground (disc) ~~exerts~~ exerts force $-\vec{F}$ on the person. This is how a person can walk (i.e. push on ground with force $\vec{F}$ $\longleftrightarrow$ ground pushes back on person with force $-\vec{F}$)

So:



$\vec{r}$ = position vector of the person at a given time t.

At this instant of walking:

1) Torque on disc (exerted by person's foot)

$$\vec{\tau}_{\substack{\text{person}\\ \text{on disc}}} = \vec{r} \times \vec{F}.$$

2) Torque on person exerted by the disc:

$$\vec{\tau}_{\substack{\text{disc}\\ \text{by person}}} = \vec{r} \times (-\vec{F})$$

Same $\vec{r}$ because both at the location of person's foot. at the point in time.

So in the system (system = person + disc) we have:

$$\vec{\tau}_{total} = \vec{\tau}_{disc} + \vec{\tau}_{person}$$
$$= \vec{r} \times \vec{F} + \vec{r} \times (-\vec{F})$$
$$= \vec{r} \times \vec{F} - \vec{r} \times \vec{F}$$
$$= 0.$$

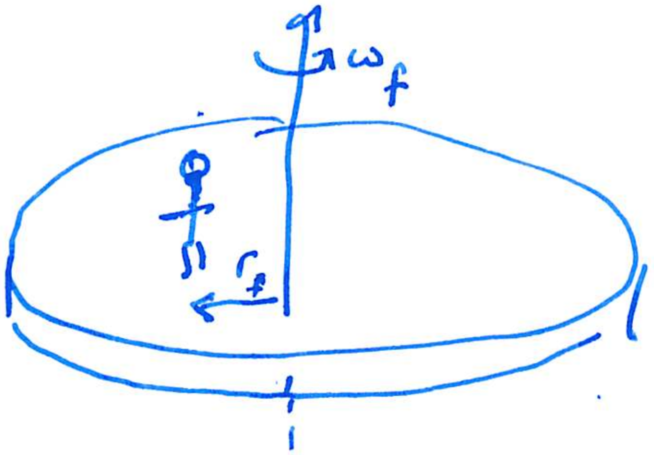
So $\vec{\tau}_{total} = 0 \Rightarrow \dfrac{d\vec{L}_{total}}{dt} = \vec{\tau}_{total} = 0.$

So: All throughout the person's motion, $\vec{\tau}_{total} = 0$.

So $\vec{L}_{total}$ always remains constant.

suppose now person stops walking.

r_{final} = final radial distance of the person from the axis of rotation.



ω_f = final angular velocity.

person + disc both rotate together at rate ω_f.

so: $\vec{L}_{\substack{total\\initial}} = \vec{L}_{\substack{total\\final}}$.

so: $[I + mr^2]\omega_i = [I + mr_f^2]\omega_f$.

$$\Rightarrow \boxed{\omega_f = \frac{[I + mr^2]\,\omega_i}{I + mr_f^2}}$$

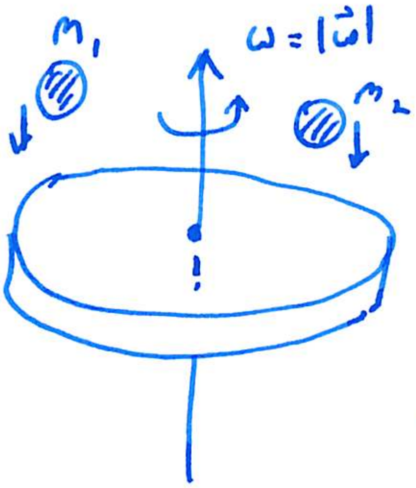
So note that if $r_f < r$; (i.e. person is closer to the axis of rotation after walking)

then $\boxed{\omega_f > \omega_i}$

(because: $I + mr^2 > I + mr_f^2$.)

☒

Ex. 2 : Another example that involves conservation of total angular momentum.

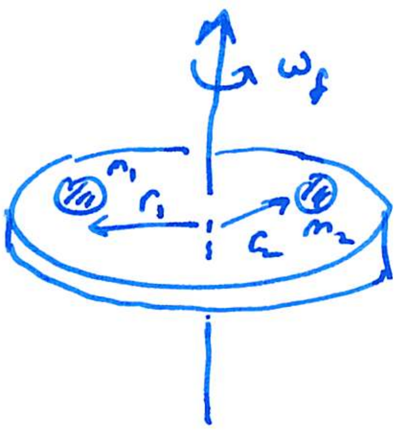


m_1 & m_2 dropped onto ~~into~~ a wheel (disc) rotating initially with angular velocity $\vec{\omega}$.

$$\vec{L}_{\substack{total\\initial}} = \underbrace{I\omega}_{\text{rotational inertia of disc}} + \underbrace{0 + 0}_{\text{initially no ~~rotational~~ angular momentum for falling balls.}}$$

After the balls land on the disc:

(they rotate with the disc) with final angular velocity $\vec{\omega}_f$.



so: $\vec{L}_{\substack{total\\final}} = I\omega_f + m_1 r_1^2 \omega_f + m_2 r_2^2 \omega_f$

$= [I + m_1 r_1^2 + m_2 r_2^2]\,\omega_f$

$\vec{L}_{\substack{initial\\~~total~~}} = \vec{L}_{\substack{total\\final}} \Rightarrow I\omega = [I + m_1 r_1^2 + m_2 r_2^2]\,\omega_f$

$$\Rightarrow \boxed{\omega_f = \frac{I\omega}{I + m_1 r_1^2 + m_2 r_2^2}}$$

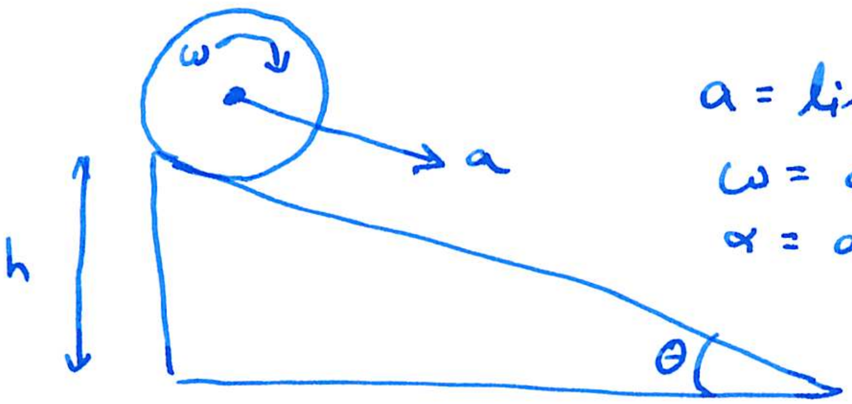
No change in $\vec{L}_{total}$ (for same reason as on pg 9)

Ex 3 An object rolls down an inclined plane <u>without slipping</u> (due to enough static friction).

M = mass of object. ; R = radius of rolling object.

Object released from rest at the top of the incline.

What is the linear acceleration of the rolling object? ~~at the bottom of the incline?~~

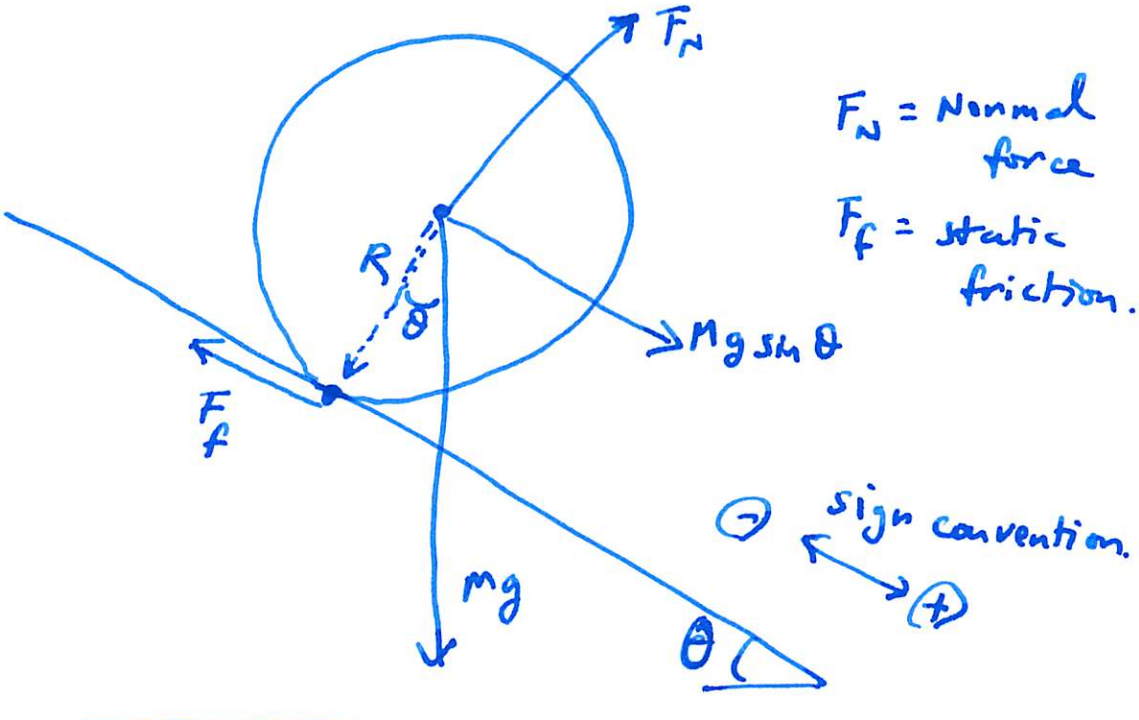


a = linear acceleration

ω = ang. velocity

α = ang. acceleration.

Solution:



F_N = Normal force

F_f = static friction.

<u>forces:</u>

$$Mg\sin\theta - F_f = Ma.$$

(Note $F_N = Mg\cos\theta$)

<u>Torque on object:</u>

$$\underbrace{F_f R}_{\substack{\text{torque on object} \\ \text{about the center of} \\ \text{the object.}}} = \underbrace{I}_{\substack{\text{rotational} \\ \text{inertia of} \\ \text{object.}}}\alpha$$

Rolling "without slipping" means : $R\omega = V$

and $R\alpha = a$

so: $F_f R = I\alpha$

$$= I\left(\frac{a}{R}\right)$$

$$\Rightarrow \quad \frac{F_f R^2}{I} = a$$

So: putting this into the force equation on (Pg 12), we have:

$$Mg\sin\theta - F_f = Ma$$

$$\Rightarrow Mg\sin\theta - F_f = \frac{MF_f R^2}{I}$$

$$\Rightarrow Mg\sin\theta = F_f\left[1 + \frac{MR^2}{I}\right]$$

$$\Rightarrow \boxed{F_f = \frac{Mg\sin\theta}{\left[1 + \frac{MR^2}{I}\right]}}$$

← friction force acting at the point of contact on the rolling object ~~frequ~~ required for rolling without slipping.

And: $a = \frac{F_f R^2}{I}$

$$= \frac{MgR^2\sin\theta}{I\left[1 + MR^2/I\right]}$$

$\Rightarrow$ Linear acceleration : $\boxed{a = \frac{MgR^2\sin\theta}{I + MR^2}}$

Note that "a" depends on the <u>shape</u> of the rolling object (e.g. sphere, disc, cylinder.) because "a" depends on "I"

← rotational inertia.

~~so~~. e.g $I_{\text{hollow cylinder}} = MR^2$

$$I_{\text{disc}} = \frac{1}{2} MR^2$$

$$I_{\text{sphere}} = \frac{2}{5} MR^2$$ ← smallest among the 3 listed here.

so: since $a = \dfrac{MgR^2 \sin\theta}{I + MR^2}$; a is largest for ~~I~~ a sphere.

so: rolling sphere reaches bottom of the incline faster than a rolling hollow sphere and a rolling disc.