Physics 1A - Oscillatory Motion

January 17, 2017

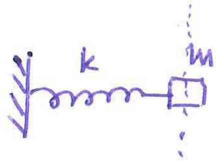
Previous time, we saw two types of equilibrium:

main difference between a stable and an unstable eq. is that the first comes with a restoring force right after we displace the system (e.g., ball).

→ Let's assume, this restoring force is directly proportional to the displacement:

$$\boxed{m\frac{d^2x}{dt^2} = -k\cdot x}$$

[Simple Harmonic Motion]



↳ solving $x(t) = C_1 \cdot \sin(w\cdot t) + C_2\cdot\cos(w\cdot t)$

$$\begin{cases} x(t) = A\cdot\cos(w\cdot t + \phi) \\ \text{with } w = \sqrt{\frac{k}{m}} \end{cases}$$

we are allowed to do this. *

* why?

From Calculus we know the following identity:

$$\cos(A+B) = \cos(A)\cdot\cos(B) - \sin(A)\cdot\sin(B)$$

why can we do: $x(t) = C_1\cdot\sin(w\cdot t) + C_2\cdot\cos(w\cdot t)$ ← which is the general solution!

? ↳ $x(t) = A\cdot\cos(w\cdot t+\phi)$

Let's use the identity:

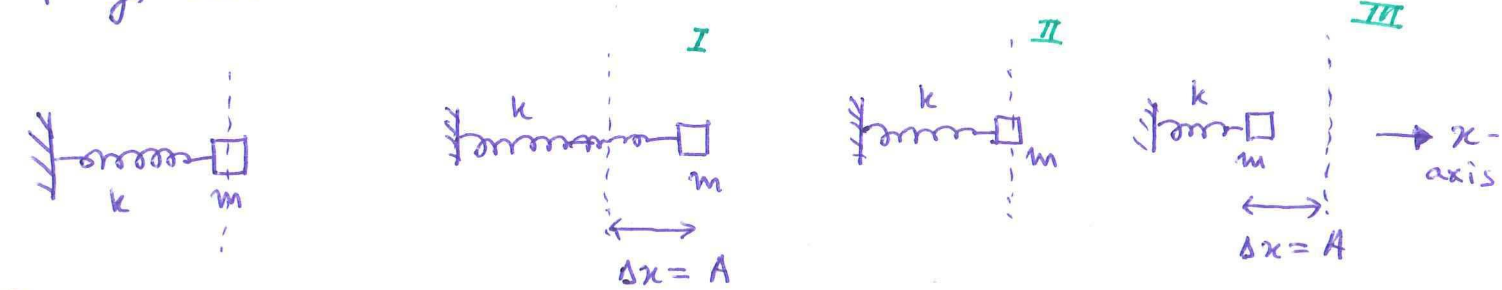
$$x(t) = A\cdot\cos(w\cdot t+\phi) = A\cdot\cos(w\cdot t)\cdot\cos(\phi) - A\cdot\sin(w\cdot t)\cdot\sin(\phi)$$

$$= \underbrace{-A\cdot\sin(\phi)}_{=C_1}\cdot\sin(wt) + \underbrace{A\cdot\cos(\phi)}_{=C_2}\cdot\cos(w\cdot t)$$

$$= C_1\cdot\sin(wt) + C_2\cdot\cos(w\cdot t)$$

So, indeed: it is allowed, given that A (amplitude) and ϕ (phase) are constants.

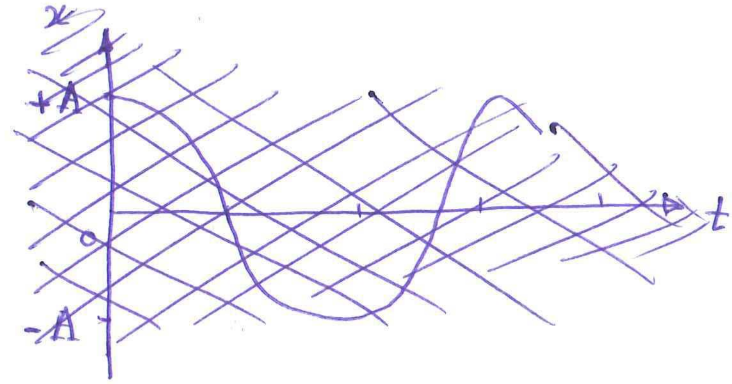
$x(t) = A \cdot \cos(\omega \cdot t + \phi)$ describes the motion of, for example, a mass m attached to a spring with spring constant k.

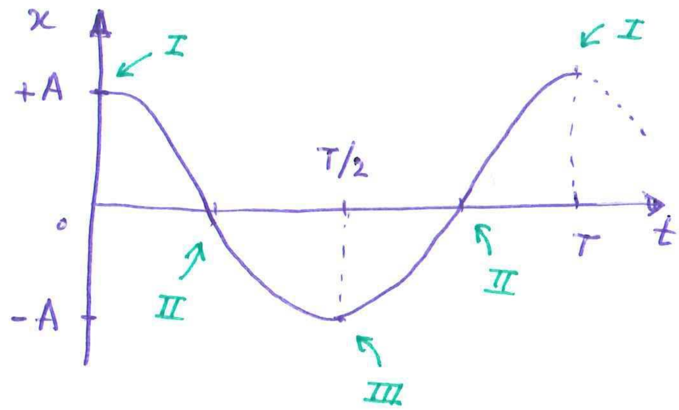


initially at equilibrium

m displaced by an amount $\Delta x = A$

Release of mass m after displacement, and $x(t)$ describes motion of m.





$x(t) = A \cdot \cos(\omega \cdot t + \phi)$:

→ the displacement is A

→ there's no friction or 'damping', decreasing A over time.

→ angular frequency is $\omega = \sqrt{\frac{k}{m}}$

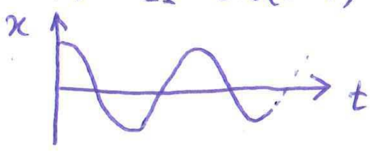
↳ period of the motion: $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

↳ frequency of the motion: $f = 1/T = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

→ the cosine-function describes the motion right after the system (for ex., mass m) is released.

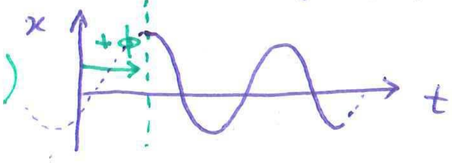
→ the phase is ϕ shifts the cosine-function in time: if $\phi > 0$, shift to the left. if $\phi < 0$, shift to the right.

$x(t) = A \cdot \cos(\omega \cdot t)$



$x(t) = A \cdot \cos(\omega \cdot t + \phi)$

$(\phi < 0)$



Today's lecture

So $x(t) = A \cdot \cos(w \cdot t + \phi)$ describes the simple harmonic motion (~~SHM~~ SHM).

Can we derive the velocity and acceleration of an object in ~~SHM~~ SHM? → YES!

diff. the argument of cos: $\frac{d(w \cdot t + \phi)}{dt} = w$

position: $x(t) = A \cdot \cos(w \cdot t + \phi)$

from diff. the cosine function

velocity: $v(t) = \frac{dx(t)}{dt} = \frac{d}{dt}\left(A \cos(w \cdot t + \phi)\right) = -A \cdot w \cdot \sin(wt + \phi)$

acceleration: $a(t) = \frac{dv(t)}{dt} = \frac{d}{dt}\left(-A \cdot w \cdot \sin(w \cdot t + \phi)\right)$

$$= -A \cdot w \cdot w \cdot \cos(wt + \phi) = -A \cdot w^2 \cdot \cos(w \cdot t + \phi)$$

With the displacement = A, what are the maximum values for the velocity and acceleration? We know that the sine-, and cosine-function oscillate between −1 and +1, so therefore:

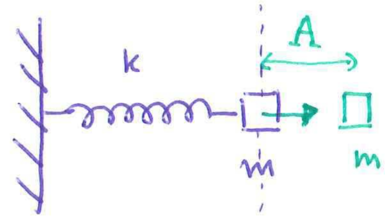
$$v_{max} = \left| \left[-A \cdot w \cdot \sin(w \cdot t + \phi) \right]_{max} \right| = A \cdot w \cdot 1 = A \cdot w$$

$$a_{max} = \left| \left[-A \cdot w^2 \cdot \cos(w \cdot t + \phi) \right]_{max} \right| = A \cdot w^2 \cdot 1 = A \cdot w^2$$

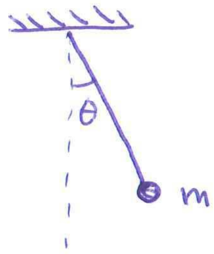
~ ~ ~ Application of ~~SHM~~ SHM ⇒ Simple Pendulum ~ ~ ~

SHM will apply to any object's motion around a stable equilibrium, as long as the restoring force is directly proportional to the object's displacement.

mass attached to spring



mass attached to string

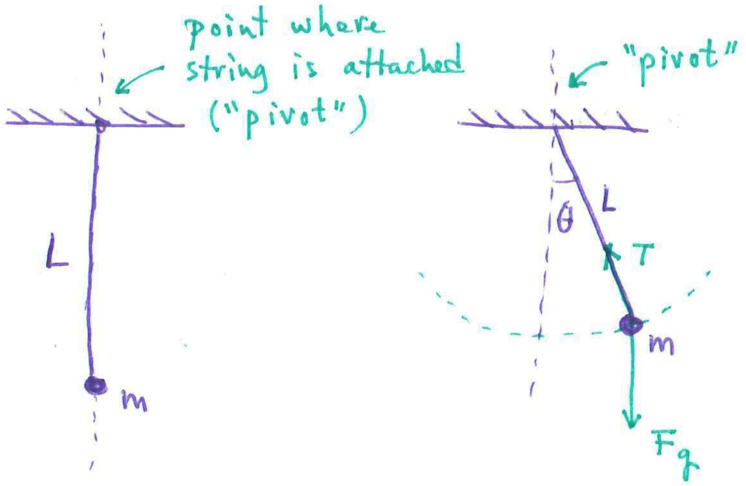


$$\begin{cases} x(t) = A \cdot \cos(w \cdot t + \phi) \\ w = \sqrt{\frac{k}{m}} \end{cases}$$

Also SHM?

A "simple" Pendulum

Let's consider a simple pendulum, which consists of a point mass m attached to a massless string of length L.



Two forces are acting on the object m:

→ gravity $F_g = m \cdot g$

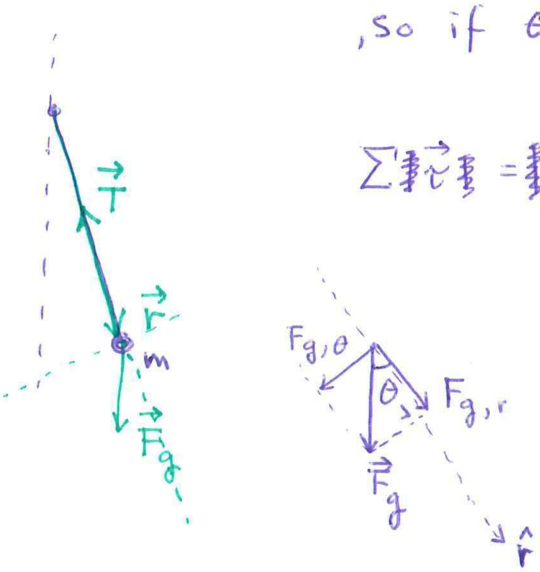
→ tension in string T

Now, if we slightly displace the mass m by an angle θ, how can we describe m's motion after we release it?

↳ Let's use torque to describe the motion of the rotating pendulum (with respect to the "pivot")

<u>Remember</u>: $|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}| \cdot |\vec{F}| \cdot \sin\theta$

, so if $\theta = 0 \rightarrow \sin(0) = 0 \rightarrow |\vec{\tau}| = 0.^*$



$$\sum \vec{\tau} = \underbrace{\vec{r} \times \vec{T}}_{=0^*} + \vec{r} \times \vec{F}_g = \vec{r} \times \vec{F}_g$$

Only $F_{g,\theta}$ will contribute *

$$|\vec{\tau}| = |\vec{r} \times \vec{F}_g| = \cancel{\vec{r} \times \vec{F}_{g,\theta}}$$

$$= -\underbrace{L}_{|\vec{r}|} \cdot \underbrace{m \cdot g}_{|\vec{F}_g|} \cdot \sin\theta$$

As the torque τ is the cause of the pendulum rotating about the equilibrium, we can use an Analog of Newton's 2nd law for oscillatory motion:

$$F = m \cdot a = m \frac{d^2 x}{dt^2} \longrightarrow \tau = I \cdot \alpha = I \frac{d^2 \theta}{dt^2}$$

(for linear motion) (for rotational motion)

So, ~~the~~ the rotational motion of the pendulum will be exactly described by the following differential equation:

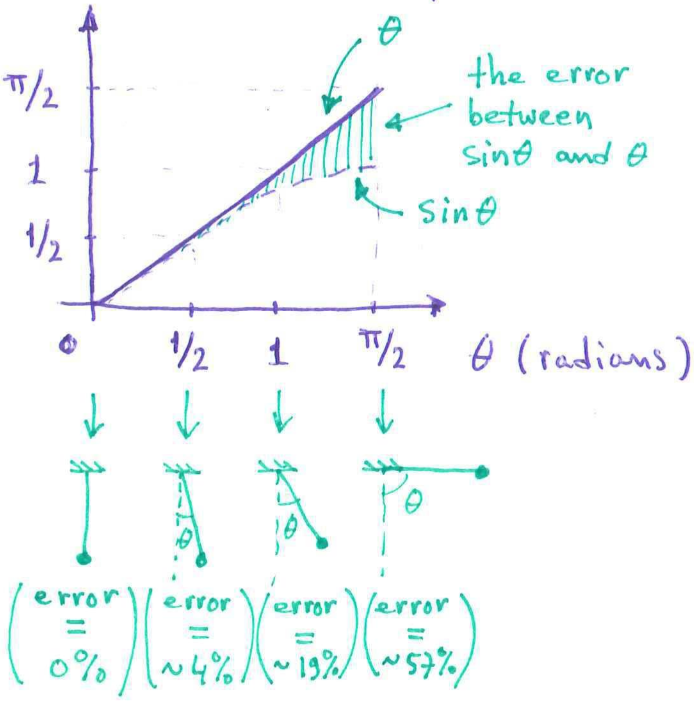
$$\boxed{I \cdot \frac{d^2\theta}{dt^2} = -m \cdot g \cdot L \cdot \sin\theta}$$

[Exact description of rotating simple pendulum]

Well, this 2nd order differential equation is actually quite hard to solve.

↳ Let's apply an approximation, which will not only make our lives easier when solving the equation, but also, make it similar to what we have seen before → SHM!

$\sin(\theta) \approx \theta$, if the angle θ adopts small values.
Drawing the two functions, $\sin(\theta)$ and θ, will show how:



We can see that for small angles θ, $\sin\theta \approx \theta$ quite well; however, as θ would approach $\pi/2$ (= 90°), the error in the approximation gets bigger.

Let's apply this approximation:

$$I \cdot \frac{d^2\theta}{dt^2} = -m \cdot g \cdot L \cdot \sin\theta \longrightarrow \boxed{I \cdot \frac{d^2\theta}{dt^2} = -m \cdot g \cdot L \cdot \theta}$$

But wait! This looks exactly like the diff. equation for the SHM.

We replaced:

$I \leftarrow m$, $\theta \leftarrow x$, $m \cdot g \cdot L \leftarrow k$.

And we know how to ~~sto~~ solve that, and what the angular frequency and period is:

$$\begin{cases} \omega = \sqrt{\dfrac{k}{m}} \\ T = \dfrac{2\pi}{\omega} \end{cases} \longrightarrow \begin{cases} \omega = \sqrt{\dfrac{m \cdot g \cdot L}{I}} \overset{I = m \cdot L^2}{=} \sqrt{\dfrac{g}{L}} \\ T = \dfrac{2\pi}{\omega} \end{cases}$$

(SHM for mass attached to spring) (SHM for simple pendulum with θ approximation)

Please <u>note</u> the following:

→ the above result shows that the frequency of the pendulum's motion is <u>not</u> dependent on the mass m!

→ we implicitly assumed that pendulum's motion is not adopting too large angles θ.

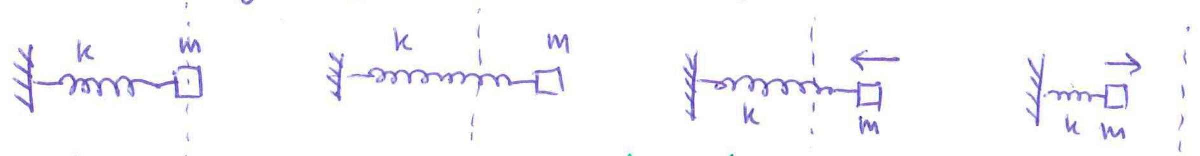
→ in fact, we can apply this result to an object of arbitrary shape that is swinging around with respect to a pivot with not too large angles θ. We only need to know: m, L, I.

Energy in SHM

→ <u>conservation of mechanical energy</u>

Let's think about SHM in terms of energy.

Your work W on displacing the mass m attached to a spring caused the oscillatory motion of the mass-spring system with respect to its stable equilibrium



equilibrium — initial displacement — KE and PE are converted into each other during the SHM.

What happens during the SHM:
Potential Energy (PE) of the stretched or compressed spring and kinetic Energy (KE) of the moving mass are converted into each continuously.

↳ PE for spring: $U = \frac{1}{2}kx^2 \underset{\substack{\uparrow \\ x = A\cdot\cos(\omega\cdot t) \\ \text{for SHM}}}{=} \frac{1}{2}\cdot k\cdot(A\cdot\cos(\omega\cdot t))^2 = \frac{1}{2}\cdot k\cdot A^2\cdot\cos^2(\omega t)$

$k = \omega^2\cdot m$

↳ KE for mass: $K = \frac{1}{2}mv^2 \underset{\substack{\uparrow \\ v = -\omega\cdot A\cdot\sin(\omega\cdot t) \\ \text{for SHM}}}{=} \frac{1}{2}\cdot m\cdot(-\omega\cdot A\cdot\sin(\omega t))^2 = \frac{1}{2}\cdot k\cdot A^2\cdot\sin^2(\omega t)$

So, with time, PE and KE will vary, as they are a function of t. But what about PE + KE?

$$U + K = \frac{1}{2}\cdot k\cdot A^2\cdot\cos^2(\omega t) + \frac{1}{2}\cdot k\cdot A^2\cdot\sin^2(\omega t)$$
$$= \frac{1}{2}\cdot k\cdot A^2\left(\cos^2(\omega t) + \sin^2(\omega t)\right) \underset{\substack{\uparrow \\ \sin^2(T)+\cos^2(T)=1}}{=} \frac{1}{2}\cdot k\cdot A^2$$

Therefore, the mechanical energy (PE + KE) is conserved (no t-dependence).

→ loss of energy in SHM: Damping

So far, we neglected friction for the pendulum's motion or any other SHM. With friction, however, will lead to energy loss resulting to a decreasing oscillation amplitude.

⇒ Often, the damping force is proportional to the object's velocity (think of air resistance, for example)

$$\underbrace{F_d}_{\text{damping force}} = -\underbrace{b}_{\text{proportionality constant}}\cdot\underbrace{v}_{\text{object's velocity}} = -b\cdot\frac{dx}{dt}$$

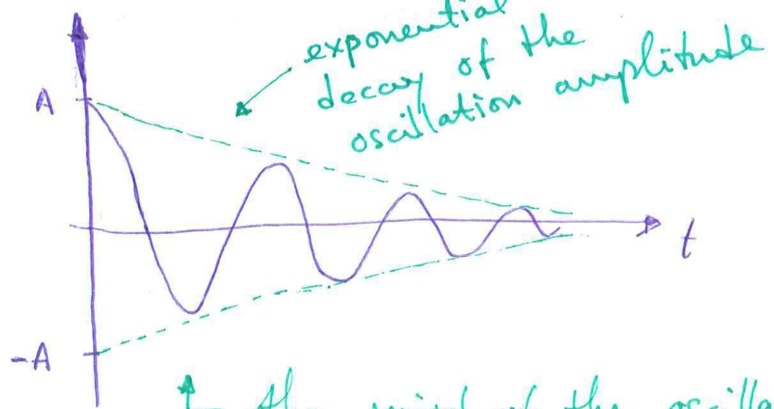
Therefore, the initial differential equation for SHM changes slightly, with an extra factor:

$$\underbrace{\underbrace{m\,\frac{d^2x}{dt^2} = -k\cdot x}_{\text{SHM without damping}} - b\cdot\frac{dx}{dt}}_{\text{SHM with damping}}$$

$$m\,\frac{d^2x}{dt^2} + b\cdot\frac{dx}{dt} + k\cdot x = 0$$

↳ The solution to this 2nd order differential equation is:

$$\boxed{x(t) = A\cdot e^{-\frac{b}{2m}\cdot t}\cdot\cos(\omega\cdot t + \phi)}$$



↳ the period of the oscillation is not changed with damping.

Physics 1A - Wave Motion

January 17, 2017

introduction:

→ Wave phenomena are ubiquitions in our physical environment. Think of:

* accoustic waves produced by a speaker reaching your ears;
* electromagnetic waves produced by a lamp reaching your eyes;
* earthquakes affecting people's lives in Groningen;

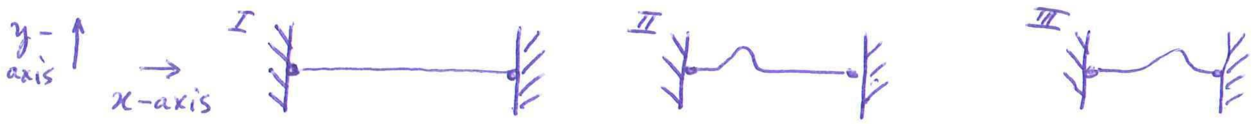
Generally speaking, waves come in two flavors:

* mechanical waves: require a medium (i.e., mass) to propagate through
* electromagnetic waves: do not require such medium.

→ definition:

"A wave is a disturbance that travels in space while carrying energy."

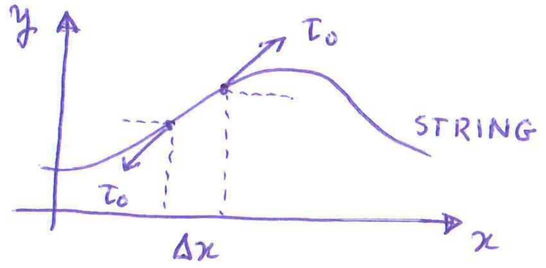
example Let's look at a string with fixed ends as bound between two walls.



- initially, the string is straight, lying along the x-axis
- if the string is locally displaced in the y-~~axis~~ direction a small amount, and quickly released, a following wave will be initiated.
- because of the continuity of the string (due to the electrostatic forces between the molecules of which the string is composed of), the originally local disturbance must spread or necessarily ~~spread~~ propagate along the string as time progresses
- and there we have our wave!

→ Wave equation

We could apply Newton's 2^{nd} law to a very small element Δx of a displaced string to find a differential equation that describes the wave.



(This is our string with mass λ_0 per unit length of the string.)

- One can show and prove the one-dimensional wave equation:

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}} \qquad \left(3D: \ \nabla^2 y = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\right)$$

- This equation describes exactly how a water or sound wave (for example) ~~with~~ propagates with wave speed v.
- Any 'reasonable' function of $x + v.t$ or of $x - v.t$ satisfies this wave equation. (d'Alembert's solution: $y(x,t) = f_1(x - v.t) + f_2(x + v.t)$)

~~~~ intermezzo // Electromagnetic Waves ~~~~

For vacuum, Maxwell summarized the theory on electric and magnetic fields (nothing to do with light or waves, you think, right?) into 4 equations:

$$\boxed{\vec{\nabla}\cdot\vec{E} = 0}, \quad \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}, \quad \boxed{\vec{\nabla}\cdot\vec{B} = 0}, \quad \boxed{\vec{\nabla}\times\vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$$

If we play a bit with these equations:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}$$

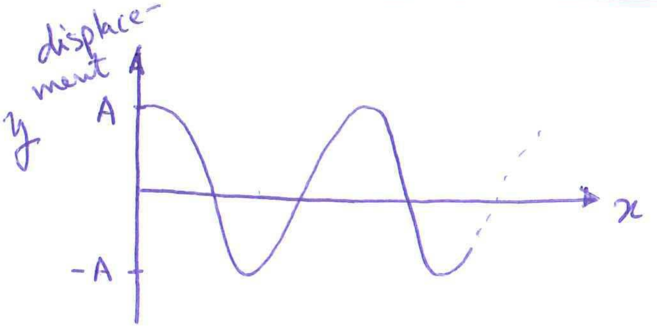
, So $\boxed{\nabla^2\vec{E} = \varepsilon_0\mu_0\frac{\partial^2\vec{E}}{\partial t^2}}$, and similarly: $\boxed{\nabla^2\vec{B} = \varepsilon_0\mu_0\frac{\partial^2\vec{B}}{\partial t^2}}$

But wait! This is the wave equation, and apparentely electromagnetic waves propagate with speed of light ($c^2 = \frac{1}{\mu_0\varepsilon_0}$)

→ Simple Harmonic wave

We saw that the solution to the (1D-) wave equation, can be any function with an argument of $x \pm v.t$

↳ Let's look at a familiar class of functions: the sinusoidal function



If we could make a "snapshot" of a sinusoidal wave at time $t=0$, and say the displacement at position $x=0$ is $+A$, then:

we have a wave that is spatially (hence the "x") sinusoidal.

$$y(x, t=0) = +A \cdot \cos(k \cdot x)$$

displacement of the wave

max. amplitude of the wave

As cos(...) cannot have a unit of, say, Meters ("x"), we need to add a constant in front to make "k.x" dimensionless (or without unit): $k \to$ "$\frac{1}{\text{meters}}$", $x \to$ "meters".

Now, if this wave ~~this wave~~ is moving in space with a constant wave speed v to the right ("−") or to the left ("+"), the wave would look exactly the same, but only shifted an amount "$v.t$" if we would look at it again a time t later.

So:

Try it yourself, by plugging in $\pm v.t$, to see we'd get the above function again.

$$y(x,t) = A \cdot \cos(k \cdot (x \pm v.t))$$

$$y(x,t) = A \cdot \cos(k \cdot x \pm k \cdot v \cdot t)$$

Let's define a few wave parameters before we continue:

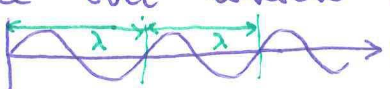
→ Amplitude: A.

The maximum value the wave adopts in its disturbance.

$A \to$ [meters]

→ Wave length: λ.

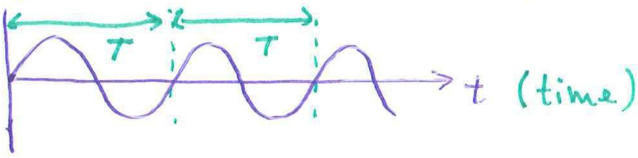
distance over which the wave pattern repeats itself



$\lambda \to$ [meters]

→ Period: T, Frequency: f

The period is the time for one (complete) oscillation.



$T \to$ [seconds]

The frequency $f = \frac{1}{T}$ (# wave oscillations per unit of time)

$f \to \left[\text{seconds}^{-1}, \text{ or } \frac{1}{\text{seconds}} \right]$

→ Angular frequency: ω

$\omega \cdot T = 2\pi \quad \to \quad \omega = \frac{2\pi}{T} \quad \left[\frac{1}{\text{seconds}} \right]$

→ Spatial frequency, or Wave number: k

$k \cdot \lambda = 2\pi \quad \to \quad k = \frac{2\pi}{\lambda} \quad \left[\frac{1}{\text{meters}} \right]$

→ Wave speed: v

$$v = \frac{\overset{\text{distance for 1 oscillation}}{\lambda}}{\underset{\text{time for 1 oscillation}}{T}} = \lambda \cdot \frac{1}{T} = \lambda \cdot f \quad \left[\frac{\text{meters}}{\text{seconds}} \right]$$

$$v = \lambda \cdot \frac{1}{T} = \frac{2\pi}{2\pi} \cdot \lambda \cdot \frac{1}{T} = \underbrace{\frac{\lambda}{2\pi}}_{\frac{1}{k}} \cdot \overbrace{\frac{2\pi}{T}}^{\omega} = \frac{\omega}{k}$$

Then, our sinusoidal wave becomes:

$$y(x,t) = A \cdot \cos(k \cdot x \pm k \cdot v \cdot t)$$

$$\boxed{y(x,t) = A \cdot \cos(k \cdot x \pm \omega \cdot t)}$$

↳ "−" represents wave going in positive x-direction.

↳ "+" represents wave going in negative x-direction.

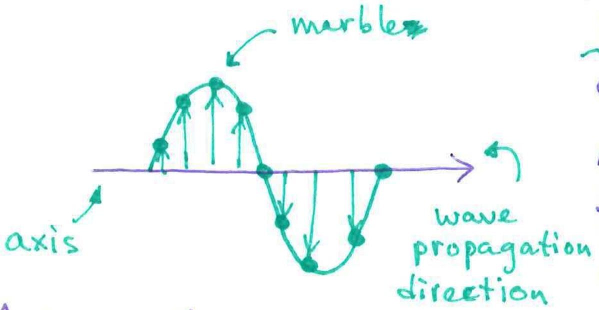
→ Wave types:

- longitudinal wave:
  wave oscillation ('disturbance') is in the same direction as the wave propagation.
- transverse wave:
  wave oscillation ('disturbance') is at right angles with respect to the wave propagation.

→ Wave Interference:

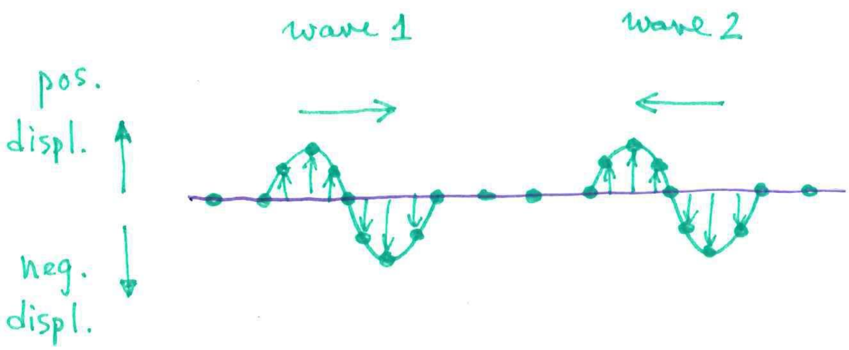
What happens if the waves would approach each other from the two opposite directions?

We say that the waves interfere, resulting into a net displacement through the superposition principle



Think of a wave consisting of interconnected marbles. At a fixed time (="snapshot") the marbles will have a displacement with respect to the axis, as indicated in the figure.

Note: the marbles on the axis do not have a displacement, and the marbles at the wave's peak have the maximum displacement.



Now, if two waves would approach each other, and as the marbles can not be displaced both in the positive as well as the negative direction, with respect to the axis, we can add the two displacements example: if wave 1 would impose a displacement on a marble +3 units (in the position direction) whereas wave 2 would impose a displacement on the same marble −1 unit (in the negative direction), then interference would result in +3 − 1 = +2 units ~~of~~ as the net displacement of that marble.

We distinguish:

→ Constructive interference, if two waves produce a larger wave displacement.

→ Destructive interference, if two waves ~~produce a smaller~~ cancel.

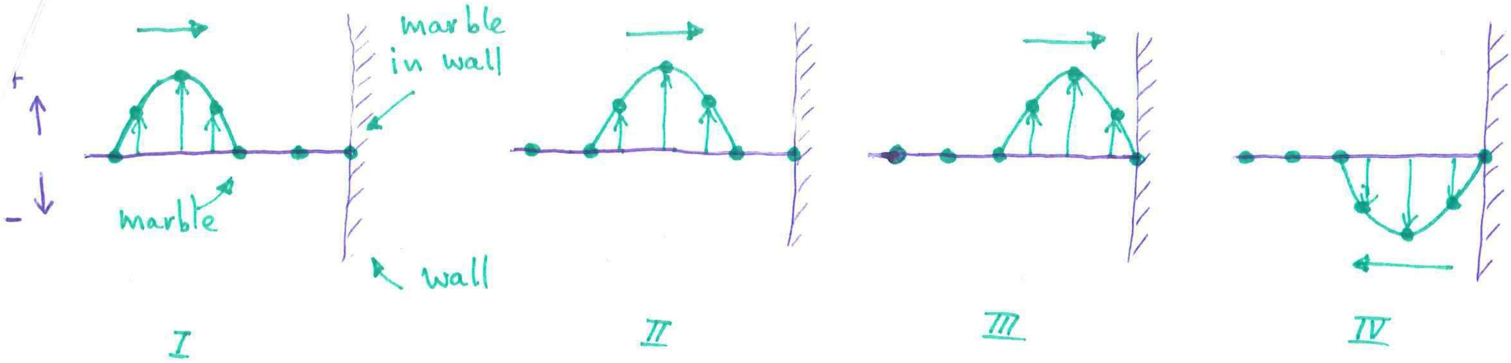
Standing Waves

In our real world, mechanical waves like sound waves do not propagate indefinitely after they are produced, as the waves will come across all sorts of barriers like a wall. What happens to the wave then?

↳ You are definitely familiar to echoes, like in a music concert hall.

⇒ Let's look at an example.
for this, we define a wall as an object that cannot be displaced.

~~wall~~



The wall is represented by the marble ("marble in wall") that cannot be moved (that is, displaced).

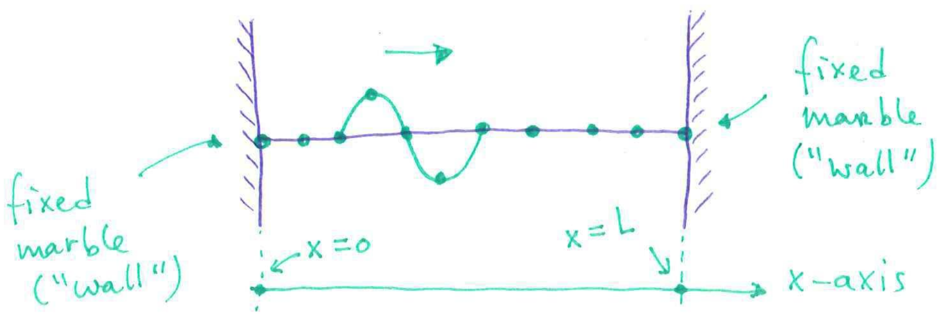
⇒ As our wave approaches the wall (I, II), the wave imposes a positive displacement on the wall ("marble in wall") as well (III)

⇒ Since the wall cannot be displaced, the wall ("marble in wall") will push the marble in the wall back; that is, it imposes a negative displacement on the marble in the wall.

⇒ Therefore, ~~the~~ the wave will be inverted (IV), and we say that the initial wave is reflected.

in general:
Reflection, transmission and refraction (wave propagation direction changes as the wave enters a second medium) are phenomena occurring at the boundary between two media.
... you will ~~cover~~ learn these topics later on...

What happens now if we restrict a transverse wave to propagate on a string with marbles, where the string ~~is~~ has fixed ("clamped") ends?



We already know that the wave will be reflected at the boundary of ~~#~~ the string (at the wall).

Suppose, we have a harmonic wave $y_1 = A \cdot \cos(kx - \omega t)$ moving from left to right. At the boundary, this wave will be reflected, so the reflected wave $y_2 = -A \cdot \cos(k \cdot x + \omega \cdot t)$ will interfere with the initial wave:

$$\begin{aligned} y(x,t) &= y_1(x,t) + y_2(x,t) \\ &= A \cdot \cos(k \cdot x - \omega \cdot t) - A \cdot \cos(k \cdot x + \omega \cdot t) \\ &= A \cdot \left(\cos(k \cdot x - \omega \cdot t) - \cos(k \cdot x + \omega \cdot t)\right) \end{aligned}$$

Using the identity:

$$\cos A - \cos B = -2 \cdot \sin\left(\tfrac{1}{2}(A+B)\right) \cdot \sin\left(\tfrac{1}{2}(A-B)\right)$$

$$\boxed{y(x,t) = 2A \cdot \sin(k \cdot x) \cdot \sin(\omega \cdot t)}$$

Note:

* the resulting wave is not propagating between the two ends ("standing")
* the amplitude of the wave is $2 \cdot A \cdot \sin(k \cdot x)$, and therefore, depending on the position between the two ends.
* as we impose that ~~the~~ the two ends are fixed ("clamped"), the following must hold:

$\rightarrow$ $x = 0 \Leftrightarrow 2 \cdot A \cdot \sin(k \cdot 0) = 0 \rightarrow \sin(0) = 0$ (confirmed)

$\rightarrow$ $x = L \Leftrightarrow 2 \cdot A \cdot \sin(k \cdot L) = 0 \rightarrow \sin(k \cdot L) = 0$

$\rightarrow k \cdot L = \underbrace{m \cdot \pi}$ $(m = 1, 2, 3, 4, \ldots)$

because $\sin(\ldots)$ is zero with argument $0, \pi, 2\pi, 3\pi, \ldots$

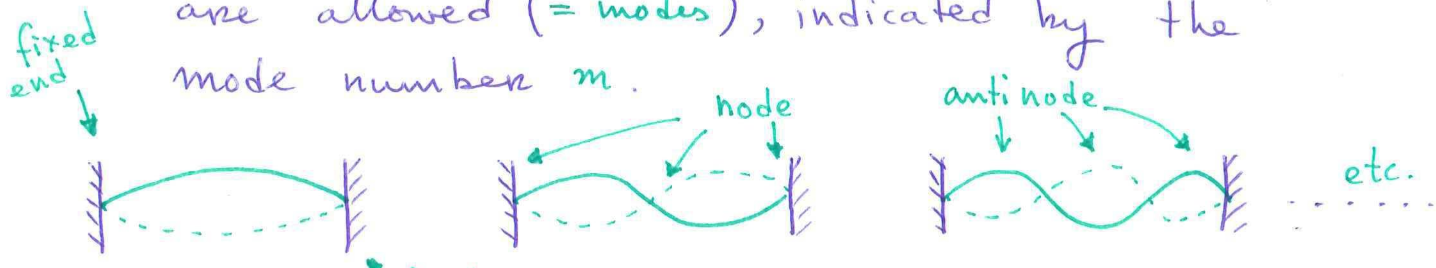
therefore:

as $k = \frac{2\pi}{\lambda}$ → remember from before

$\frac{2\pi}{\lambda} \cdot L = m \cdot \pi$ → (i) $\boxed{L = m \cdot \frac{\lambda}{2}}$ standing wave between two fixed ends (or two open ends)

The condition (i) above valid for a standing wave between two fixed ends a distance L from each other.

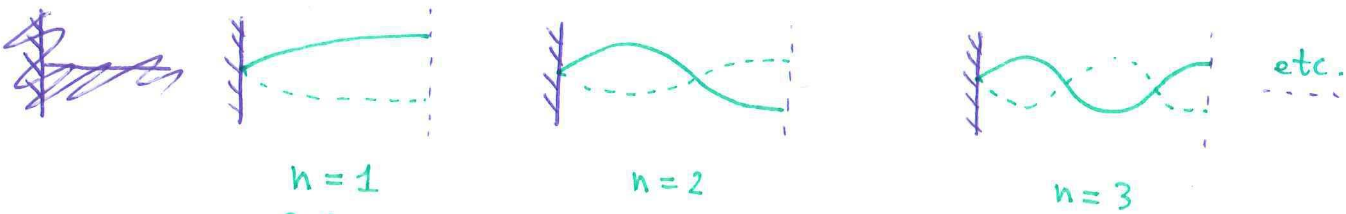
→ It says that between the ends only a discrete set of waves with wavelengths of $\frac{\lambda}{2}$, λ, $\frac{3 \cdot \lambda}{2}$, ... are allowed (= modes), indicated by the mode number m.



m = 1	m = 2	m = 3
$\lambda/2$	λ	$3 \cdot \lambda/2$
"fundamental"	"1st higher mode"	"2nd higher mode"

→ We understand the idea, right?: between two fixed (or two open) ends ~~the~~ we can "fit" discrete, integer sets of $\lambda/2$.

If we would have restricted the wave the move between one fixed and one open end, then:



n = 1	n = 2	n = 3
$\lambda/4$	$3 \cdot \lambda/4$	$5 \cdot \lambda/4$

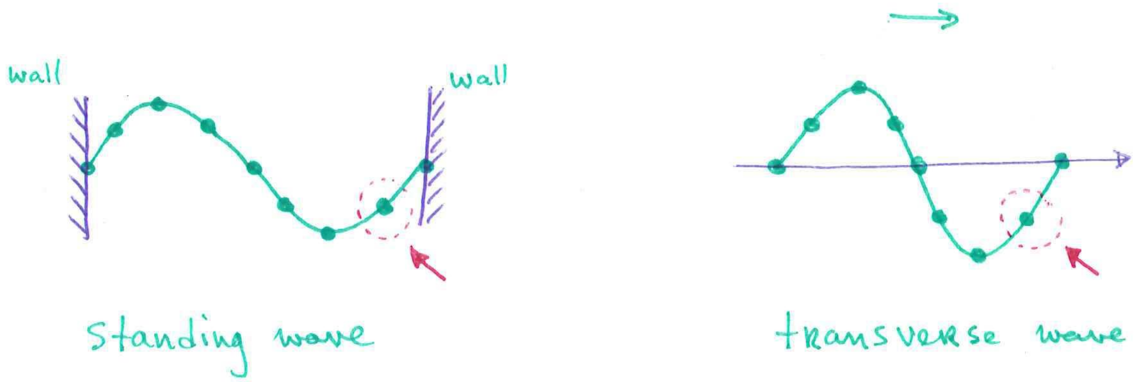
Similarly, we can derive the condition that must be satisfied for a standing wave between a fixed and an open end

(ii) $\boxed{L = (n+1) \cdot \frac{\lambda}{4}}$ standing wave between a fixed and an open end.

Transverse wave VS Standing wave

Can you reason your answer to the following question?

Is there a difference between displacements of the indicated marble, in time?



(H)