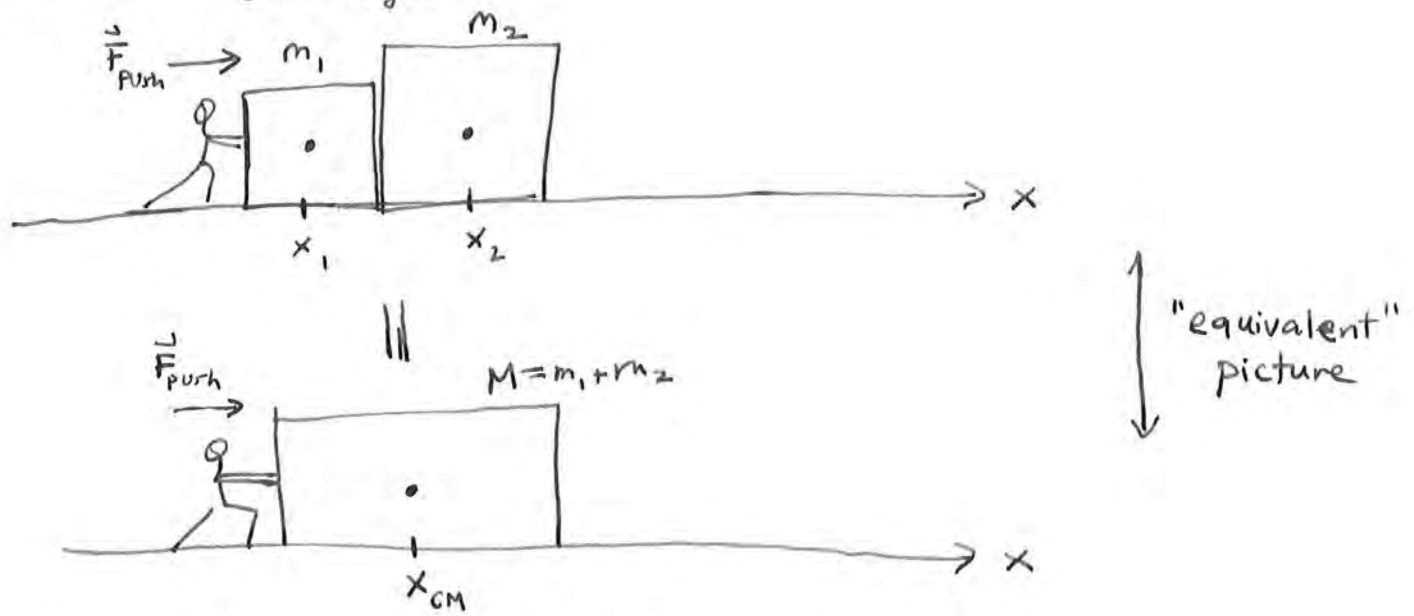
Lecture 6 : Center of mass (continued) and conservation of linear momentum

- Yesterday, we considered the example of pushing 2 blocks together to motivate the definition of center of mass.

Recall from yesterday:



↑ position of "center of mass"

The main idea is that we can reduce a complex system of many blocks, particles, ~~atoms~~ atoms down to a picture consisting of just one object ← the "center of mass"

In the above example, we have:

$$\vec{F}_{push} = (m_1 + m_2)\vec{A}_{cm} \quad \leftarrow \text{both blocks accelerate together.}$$

↑ total (net) external force (external force means something outside the system exerts force on the system) Here, the system ~~is the 2 block~~ consists of the 2 blocks.

And if we write

$$\vec{F}_1 = m_1 \vec{A}_1 \qquad \vec{F}_2 = m_2 \vec{A}_2$$

↑ total external force on m_1 ↑ total external force on m_2

(In our example, $\vec{A}_1 = \vec{A}_2 = \vec{A}_{cm}$, but we can still write the Newton's 2^{nd} law equation for each block this way.)

So:
$$m_1 \vec{A}_1 + m_2 \vec{A}_2 = (m_1 + m_2)\vec{A}_{cm}$$

$$\Rightarrow \quad m_1 \frac{d\vec{v}_1}{dt} + m_2 \frac{d\vec{v}_2}{dt} = (m_1+m_2)\frac{d\vec{V}_{cM}}{dt}$$

$\vec{V}_{cM}$ = velocity of center of mass
$\vec{v}_1$ = velocity of block 1
$\vec{v}_2$ = velocity of block 2

$$\Rightarrow \quad \frac{d}{dt}\left(m_1\vec{v}_1 + m_2\vec{v}_2\right) = (m_1+m_2)\frac{d\vec{V}_{cM}}{dt}$$

$$= \frac{d}{dt}\left((m_1+m_2)\vec{V}_{cM}\right)$$

Two sides of above equation are equal to each other if

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{V}_{cM}$$

Now, $\vec{v}_1 = \frac{dx_1}{dt}$ $\quad \vec{V}_2 = \frac{dX_2}{dt}$ $\quad \vec{V}_{cM} = \frac{dx_{cM}}{dt}$

So:

$$m_1\frac{dx_1}{dt} + m_2\frac{dx_2}{dt} = (m_1+m_2)\frac{dX_{cM}}{dt}$$

$$\Rightarrow \quad \frac{d}{dt}\left(m_1x_1 + m_2x_2\right) = \frac{d}{dt}\left((m_1+m_2)X_{cM}\right)$$

Above, two sides of equation are equal to each other if

$$m_1x_1 + m_2x_2 = (m_1+m_2)X_{cM}.$$

$$\Rightarrow \quad \boxed{X_{cM} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}}$$

↑ position of center of mass.

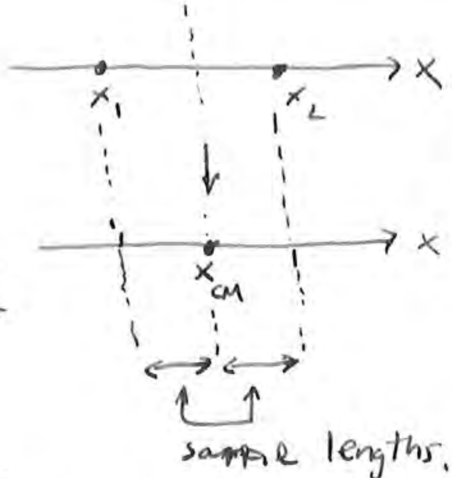
<u>Remark</u>: strictly speaking, since a derivative of a constant $=0$, we still satisfy $m\vec{v}_1 + m\vec{v}_2 = (m_1+m_2)\vec{V}_{cM}$ if we have
$$m_1\vec{x}_1 + m_2\vec{x}_2 = (m_1+m_2)\vec{x}_{cM} + \text{Constant}.$$
But since we're <u>choosing</u> to define the center of mass the way we like it, we choose the constant $=0$.

~~In general, if we have N particles~~

Note that if $m_1 = m_2$:

$$X_{CM} = \frac{m_1 (x_1 + x_2)}{2m_1}$$

$$= \frac{x_1 + x_2}{2}$$ ← Average of the two positions



If $m_1 < m_2$:

then X_{CM} is closer to the position of m_2 than it is to the position of m_1

(To see this, note that as we take $\underset{\text{limit}}{m_1 \to 0}$, we have

$$X_{CM} = \lim_{m_1 \to 0} \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$= \frac{m_2 x_2}{m_2}$$

$$= x_2 .$$

In general, if we have N particles (blocks, atoms, people, etc.) which may or may not be touching one another, we define the position of center of mass to be:

(for 1-dimensional motion):

$$X_{CM} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_N x_N}{m_1 + m_2 + \cdots + m_N}$$

$$= \frac{\sum_{i=1}^{N} m_i x_i}{\text{total mass of system}}$$

x_i = position on x-axis of i^{th} particle

m_i = mass of i^{th} particle

Here, "system" consists of N ~~total mass~~ particles.

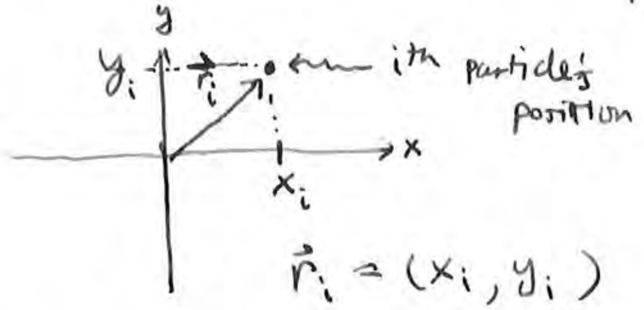
For 2 and 3 dimensional motion and objects:

We define the center of mass as an "imaginary" particle whose position is at $\vec{r}_{CM}$:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \dots + m_N\vec{r}_N}{m_1 + m_2 + \dots + m_N}$$

where

$\vec{r}_i$ = position of i^{th} particle



$\vec{r}_i = (x_i, y_i)$

Since we can consider each x-, y-, and z-component of $\vec{r}_i$ separately, we get:

$$\vec{r}_{CM} = (x_{CM}, y_{CM}, z_{CM})$$

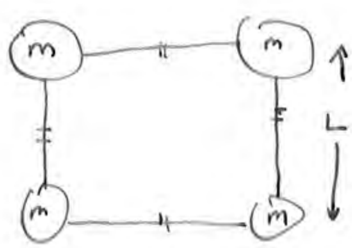
$$x_{CM} = \frac{m_1x_1 + m_2x_2 + \dots + m_Nx_N}{m_1 + m_2 + \dots + m_N}$$

$$y_{CM} = \frac{m_1y_1 + m_2y_2 + \dots + m_Ny_N}{m_1 + \dots + m_N}$$

$$z_{CM} = \frac{m_1z_1 + m_2z_2 + \dots + m_Nz_N}{m_1 + \dots + m_N}$$

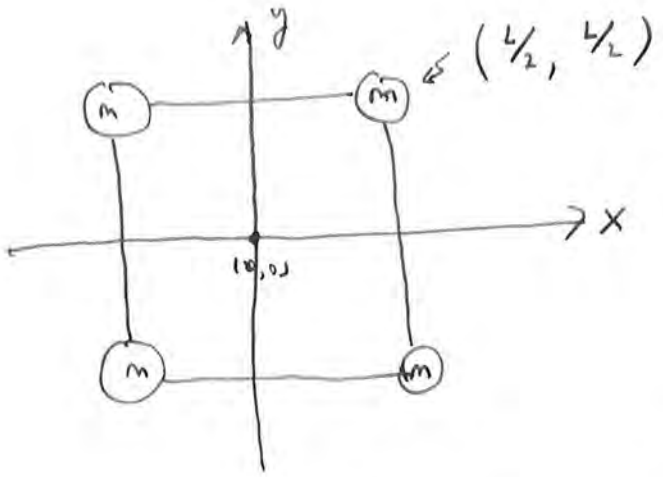
You can calculate the position of CM component by component.

Ex:



Square of side length L.
4 equal masses at corners of the square.
Where's the center of mass?

sol'n: We need to first pick an origin so that we can assign position to each particle.
You can pick the origin to be anywhere. But let's pick the origin to be at the center of the square.

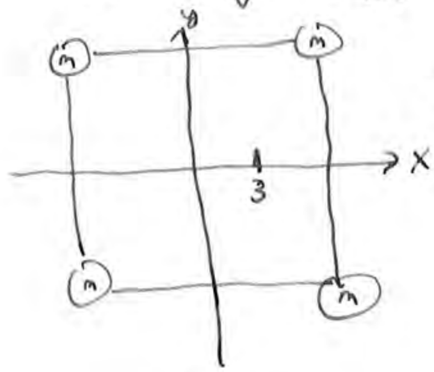


We can use the formula for the center of mass, but here, we can use symmetry to quickly get the answer.

We want $\vec{r}_{cm} = (x_{cm}, y_{cm})$

• First, calculate x_{cm}: From symmetry, we can see that $x_{cm} = 0$.

Because if it weren't, then x_{cm} would have to be either (+) or (−). Say x_{cm} is some (+) number, like $x_{cm} = 3$.



← This looks wrong. You could say that if $x_{cm} = 3$ is a valid answer, then so is $x_{cm} = -3$, because the system on the right side of y-axis looks the same as the system on the left side of y-axis.

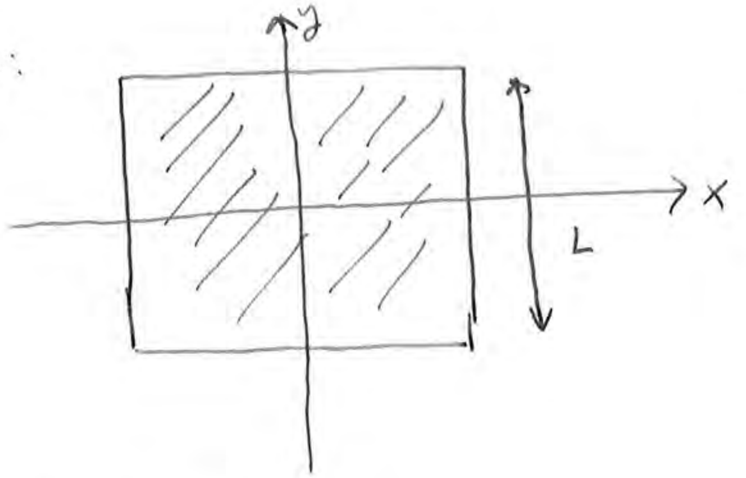
So we must have $x_{cm} = 0$

• $y_{cm} = 0$: for same reason as above.

So $\boxed{\vec{r}_{cm} = (0, 0)}$ ← If you use the formula from previous page, you get the same answer.

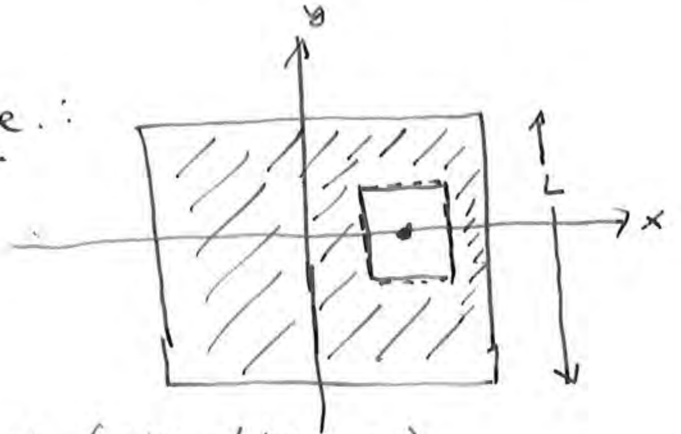
☒

Ex:



- 2-dimensional square of total mass M. side length L.
- Uniform mass density.

Now, ~~take out~~ make a square hole:



The square hole has side length "a"

$(a < \frac{L}{2})$

and the hole's center is at $(x, y) = (L/2, 0)$.

Qu: Where's the center of mass of this new square with the hole?

Sol'n: Again, we want $\vec{r}_{cm} = (x_{cm}, y_{cm})$.

- Let's again solve for x_{cm} and y_{cm} component-by-component.

- y-component: By symmetry, $y_{cm} = 0$ because the image above the x-axis looks the same as the image below the x-axis.

- X-component: We know that in the full square (before the hole introduced), the CM is at the center of the square $(0, 0)$.

(i.e. If we let x_{full} = C.M. x-component of full square)

then $x_{full} = 0$.

The full square is a system that consists of

↑ filled hole (the little square that was taken out

so
$$M x_{full} = \frac{M_{\boxdot}\, x_{cm,\boxdot} + M_{\blacksquare}\, x_{\blacksquare,cm}}{M}$$

where $x_{cm,\boxdot}$ ← ~~cm~~ CM that we want ($= x_{cm}$)

$x_{\blacksquare,cm}$ ← cm of the little square that was taken out.

Uniform mass distribution ⇒ $M_{\blacksquare} = \rho\cdot(\text{Area of } \blacksquare)$
$$= \rho a^2 \qquad \rho = \frac{M}{L^2}$$
$$= \frac{M a^2}{L^2}$$

$$M_{\boxdot} = M - M_{\blacksquare}$$
$$= M\left(1 - \frac{a^2}{L^2}\right)$$

And, $x_{\blacksquare,cm} = L/2$.

$x_{full} = 0$.

so we have:
$$0 = \frac{\cancel{M}\left(1-\frac{a^2}{L^2}\right)x_{cm} + \frac{\cancel{M}a^2}{L^{\cancel{2}}}\cdot\frac{\cancel{L}}{2}}{\cancel{M}}$$

$$\Rightarrow \quad -\frac{a^2}{2L} = \left(1 - \frac{a^2}{L^2}\right)x_{cm}$$

$$\Rightarrow \quad x_{cm} = \frac{-a^2}{2\cancel{L}}\,\frac{L^{\cancel{2}}}{L^2-a^2}$$

$$\Rightarrow \quad \boxed{x_{cm} = \frac{-a^2 L}{2(L^2-a^2)}}$$

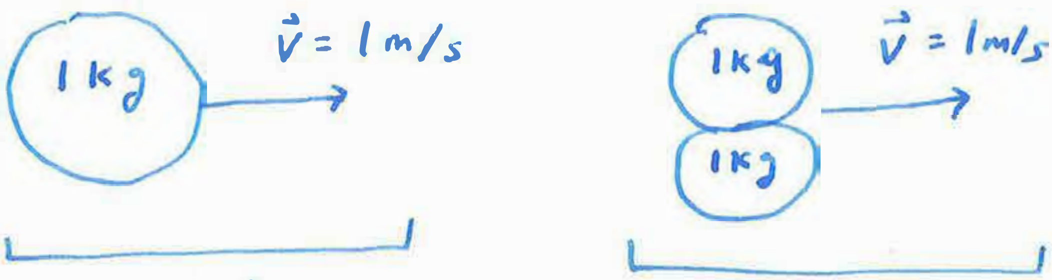
so $$\boxed{\vec{r}_{cm} = \left(\frac{-a^2 L}{2(L^2-a^2)},\ 0\right)}$$

∎

5.2. Momentum (Linear momentum)

Momentum = Quantity of motion.

Linear momentum = Quantity of linear motion



A B

We want to define momentum so that B has twice the quantity of motion (momentum) as A.

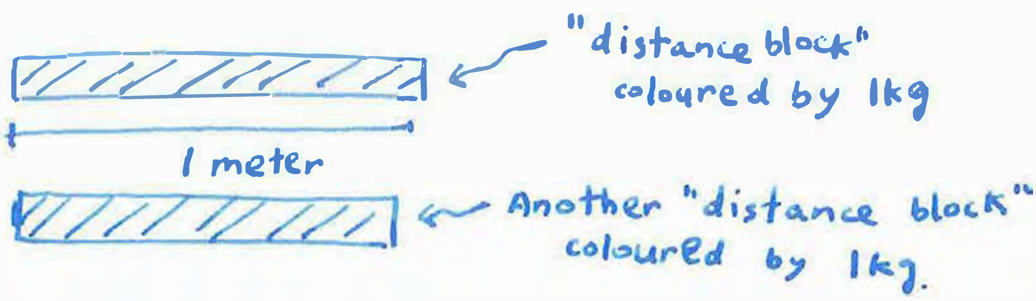
C D

Want to define momentum so that D has two times more momentum than C.

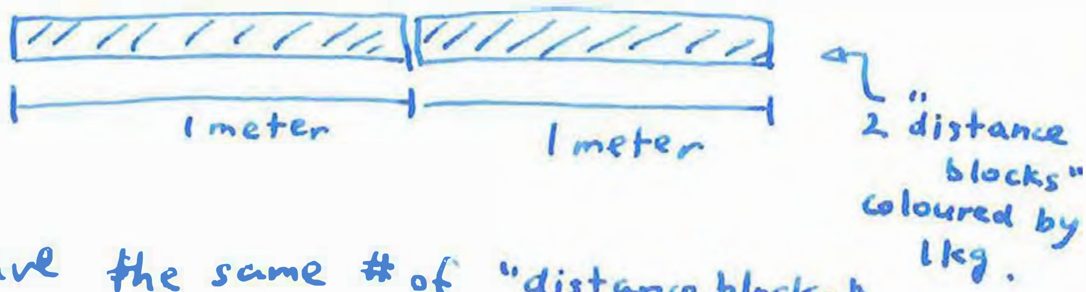
And we want: B and D have the same momentum ("quantity of motion").

<u>Motivation for this</u>: In 1 second, we have:

<u>In B</u>:



<u>In D</u>:



✱ B & D have the same # of "distance blocks" coloured in 1 second.

[<u>Note</u>: In reality, the blocks don't actually "colour" anything when they move; and the concept of "distance blocks" is a fictional one. But I introduce them here to help you intuitively and visually understand momentum.]

· We define linear momentum to be:
<u>(Motivated by above and reasons on previous page)</u>

$$\boxed{\vec{P} = m\vec{v}}$$

← Now 1.) B and D have same momentum.
2.) B has $2 \times \vec{P}$ than A
3.) D has $2 \times \vec{P}$ than C.

Note that $\vec{p}$ is a vector

(has magnitude and direction).

This is because we want to distinguish between

$\vec{p} = 1$ kg·m/s $\qquad\qquad$ $\vec{p} = -1$ kg·m/s

So momentum ($\vec{p}$) is a vector: It has direction & magnitude.

The actual definition of force $\vec{F}$ is: $\boxed{\vec{F} = \frac{d\vec{p}}{dt}}$ ← This is how Newton originally defined force.

Note that if you have an object of mass m that doesn't change over time, (i.e. m = constant)

then we have: $\vec{F} = \frac{d\vec{p}}{dt}$

$= \frac{d}{dt}(m\vec{v})$

$= m\frac{d\vec{v}}{dt}$

$= m\vec{a}$ → So we retrieve the familiar $\vec{F} = m\vec{a}$.

↑ Acceleration

> So force is just the rate of change of linear momentum.

↑ Definition of force

But if the mass of object changes over time (e.g., as particle is moving in the air, it has dust particles sticking on it (i.e. particle gains mass.))

then we would have: $\vec{F} = \frac{d\vec{p}}{dt}$

$= \frac{d(m\vec{v})}{dt}$

$= \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt}$ (by the "chain rule" of differentiation.)

$$\Rightarrow \boxed{\vec{F} = \vec{v}\frac{dm}{dt} + m\vec{a}}$$

Ex 10

1-D motion
2 particles on collision course.

The total momentum of the system (composed of these 2 particles) is:

$$\vec{P}_{total} = \underbrace{m_1\vec{V}_1}_{\text{momentum of } m_1} + \underbrace{m_2\vec{V}_2}_{\text{momentum of } m_2}$$

Note that since we have

$$\vec{V}_{CM} = \frac{m_1\vec{V}_1 + m_2\vec{V}_2}{m_1+m_2}$$

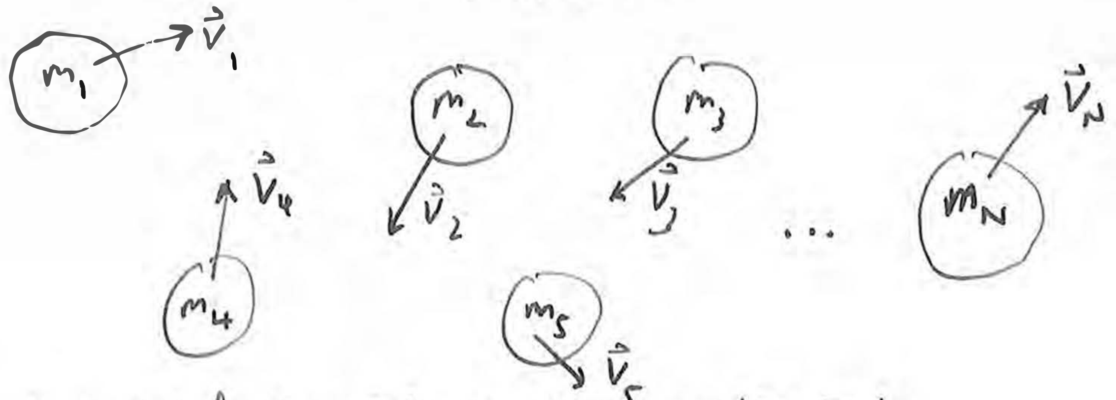
velocity of center of mass (CM)

we have:

$$\boxed{m_{total}\vec{V}_{CM} = \vec{P}_{total}}$$

$(m_{total} = m_1 + m_2)$ ☑

Ex 11 Now consider a system of N particles $(N \geq 2)$ that can move in 3-dimensions.



Then the total momentum of the system is:

$$\vec{P}_{total} = m_1\vec{V}_1 + m_2\vec{V}_2 + \ldots + m_N\vec{V}_N$$
$$= \sum_{j=1}^{N} m_j\vec{V}_j$$

But just as in ~~the~~ previous example, we note that:

$$\vec{V}_{CM} = \frac{\sum_{j=1}^{N} m_j\vec{v}_j}{m_{total}} \qquad m_{total} = \sum_{j=1}^{N} m_j$$

$$\Rightarrow m_{total}\vec{V}_{cm} = \sum_{j=1}^{N} m_j\vec{v}_j$$
$$= \vec{P}_{total}.$$

$\cancel{m_{total}\vec{V}_{cm} = \vec{P}_{total}}$

$$\boxed{m_{total}\vec{V}_{CM} = \vec{P}_{total}}$$

☑

The previous example is showing us a <u>remarkable result</u>!

Recall that we first learned that the center of mass (CM) is a point that represents a system of many particles.

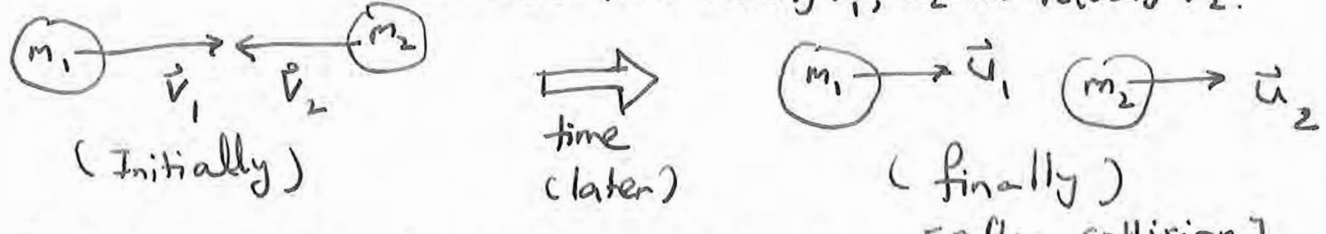
It has the same mass as the total mass of the system. In addition, we now see from the previous example that the CM's momentum represents the total momentum of the system as well.

So in many aspects, the CM is a point that truly represents the entire system (which may be composed of many particles moving in complicated ways.)

☒

Collisions between objects

Ex 1. 2 particles, masses m_1 & m_2, collide with each other head on (i.e. they're directly heading towards each other.)
Initially, m_1 has velocity $\vec{v}_1$, m_2 has velocity $\vec{v}_2$.



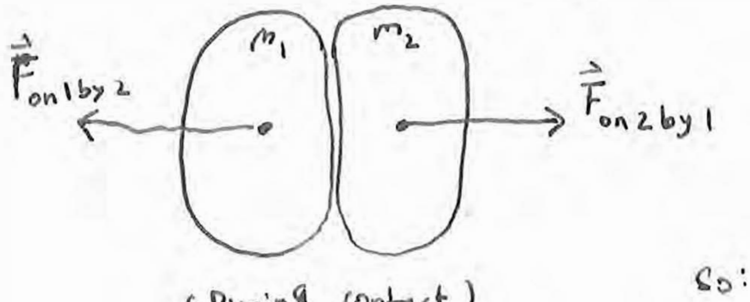
After they collide, m_1 moves ~~with~~ with velocity $\vec{u}_1$ and m_2 moves with velocity $\vec{u}_2$.

Qu: Determine $\vec{u}_2$ in terms of all other parameters of the system (i.e. $m_1, m_2, \vec{v}_1, \vec{v}_2, \vec{u}_1$).

Sol'n: Initially, (before collision); $\vec{P}_{\substack{total\\before}} = m_1\vec{v}_1 + m_2\vec{v}_2$
← total momentum of system before collision.

After the collision, the system's total momentum is: $\vec{P}_{\substack{tot.\\after}} = m_1\vec{u}_1 + m_2\vec{u}_2$.

Now, the key is to analyze what force each particle feels during the collision (i.e. while the particles are in contact with each other during the brief amount of time.)



By Newton's 3rd law,

$$\vec{F}_{on\,1\,by\,2} = -\vec{F}_{on\,2\,by\,1}$$

So: $\vec{F}_{Net} = \vec{F}_{on\,1\,by\,2} + \vec{F}_{on\,2\,by\,1}$
$= 0$

Net (total) force in the system.

And note that $\frac{d\vec{P}_{total}}{dt} = \vec{F}_{net}$; where $\vec{P}_{total}$ = total momentum of system at any time

- During the collision, we have $\frac{d\vec{P}_{total}}{dt} = 0$.
- So, this means that $\vec{P}_{\substack{total \\ before}} = \vec{P}_{\substack{total \\ after}}$ [i.e. The total momentum ~~been~~ never changes during the collision.]

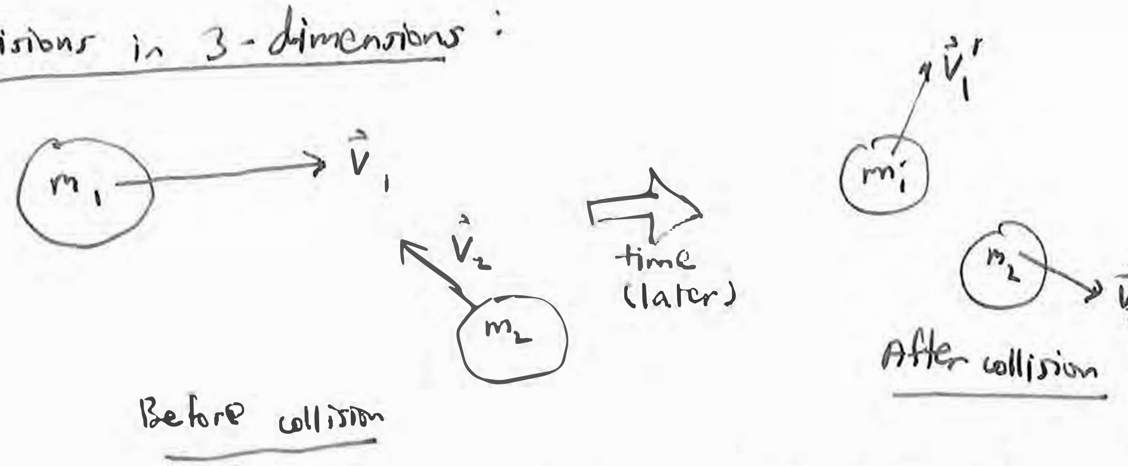
Thus: $m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{u}_1 + m_2\vec{u}_2$

$$\Rightarrow \boxed{\frac{m_1(\vec{v}_1 - \vec{u}_1) + m_2\vec{v}_2}{m_2} = \vec{u}_2}$$ ← velocity of m_2 after the collision, expressed in terms of all other variables. ☒

In the previous example, we constrained ourselves to 1-dimension (head-on collisions). But the same reasoning should also work in 3-dimensional motion (i.e. Not just head-on collisions.)

Let's repeat the reasoning:

Ex 2 Collisions in 3-dimensions:

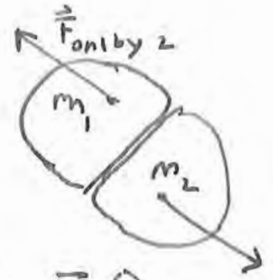


- Before collision: $\vec{P}_{total} = m_1\vec{v}_1 + m_2\vec{v}_2$
  ↑ total momentum of system
- After collision: $\vec{P}'_{total} = m_1\vec{v}_1' + m_2\vec{v}_2'$

Note that: 1.) Before collision: $\frac{d\vec{P}_{total}}{dt} = \vec{F}_{total} = 0$ ← No net force in system because no force on either particle

2.) After collision: $\frac{d\vec{P}'_{total}}{dt} = \vec{F}'_{total} = 0$ ←

And 3) During the Collision:



$\vec{F}_{on\,1\,by\,2} = -\vec{F}_{on\,2\,by\,1}$ ← Newton's 3rd law.

so $\dfrac{d\vec{P}_{total\,(during\,collision)}}{dt} = \vec{F}_{net\,(during\,collision)} = 0.$

(because $\vec{F}_{net\,(during\,collision)} = \vec{F}_{on\,1\,by\,2} + \vec{F}_{on\,2\,by\,1} = 0$)

So putting 1.), 2.) & 3.) together, we see that at no point during the whole process (before, during, and after collision), is there ever any net (total) force in the system.

∴ Total momentum remains constant during the whole process.

We say that "the total momentum is conserved during collision"

Thus: $m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$

EX3: In the previous example, (a) What is the velocity of the Center of mass (CM) before the collision?

(b) After the collision?

(c) How does the position of CM changes over time?

Sol'n: (a) Before the collision: $\boxed{\vec{V}_{CM} = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1+m_2}}$

(b). After the collision: $\boxed{\vec{V}_{CM}' = \dfrac{m_1\vec{v}_1' + m_2\vec{v}_2'}{m_1+m_2}}$

CM velocity before and after collision.

Important: Note that since $\dfrac{d\vec{P}_{total}}{dt} = 0$ at all times, we have

$$m_{total}\vec{V}_{CM} = m_{total}\vec{V}_{CM}'$$

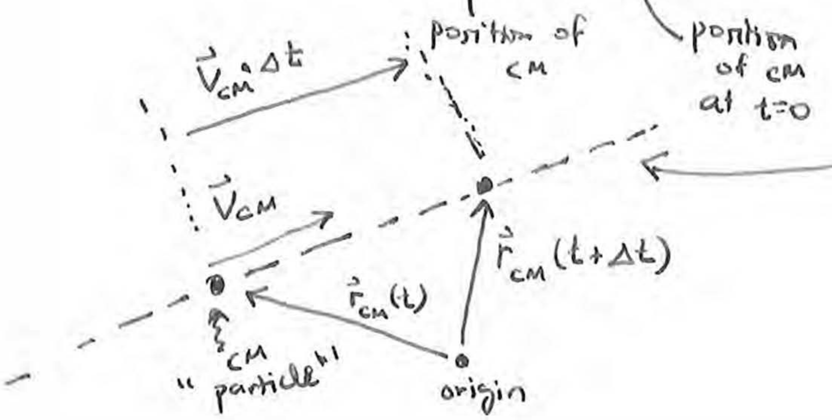
$$\Rightarrow \boxed{\vec{V}_{CM} = \vec{V}_{CM}'}$$

(due to conservation of total momentum)

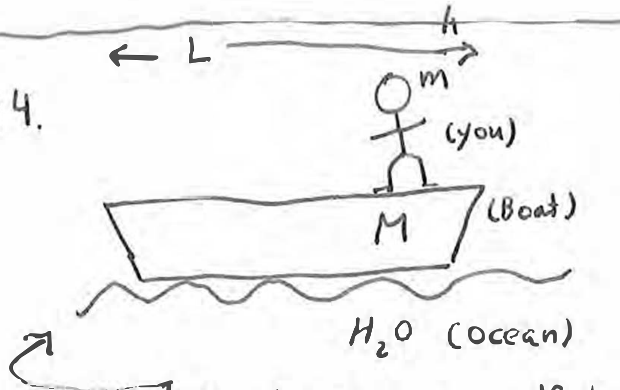
(c) How does the CM's position change over time?

sol'n: from (b) & (a) [due to the conservation of the total momentum]
we see that the CM (point-particle that represents the whole system.)
moves at all times with constant velocity $\vec{V}_{cm}$ $(= \vec{V}'_{cm})$.

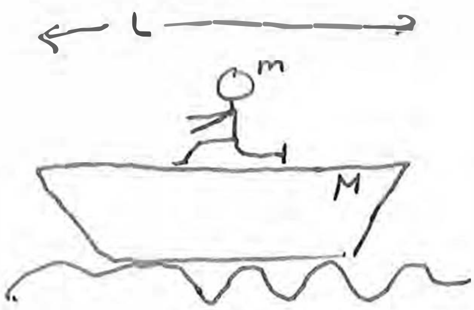
so: $$\boxed{\vec{r}_{cm}(t) = \vec{r}_{cm}(0) + \vec{V}_{cm} \cdot t}$$ (constant velocity motion in a straight line)



Ex 4.



~~Init~~ Initially: Everyone at Rest.



Initially, you and the boat are at rest on a calm ocean water.

Assume there's no friction between the ocean and boat, and between you and the boat. You stand at the end of the boat at $t=0$. Boat has length L.

You start to walk to the ~~left end~~ left end of the boat. Sometimes you walk at a fast speed, sometimes you walk at a slower speed.
When you reach the left end of the boat, you stop.

Questions
(A) What's happening to the boat as you're walking?
(B) What happens to the boat when you stop moving after reaching the left end of the boat?
(C) Does the boat move at all? If so, by how much and in which direction?

sol'n (A) It might seem like this is a challenging question because the person is moving with a speed that's varying over time (sometimes walking fast, sometimes slow.)

But this question becomes easy if you just think about the main principles that are underlying the phenomenon here.

The main principles here are: 1) Internal forces add up to zero (i.e. they cancel each others)

↳ covered in the previous lecture.

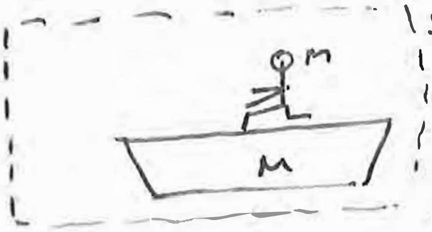
and 2) $\frac{d\vec{P}_{total}}{dt} = \vec{F}_{net}$. (And $\vec{F}_{net} = \vec{F}_{external} + \vec{F}_{internal}$)

$\vec{F}_{external}$: total external force $= 0$

$\vec{F}_{internal}$: total internal force $= 0$

Covered in previous lecture.

Note that $\vec{F}_{net} = 0$ here because there are no external forces in the system



System = you + boat.

Here, internal forces are: $\vec{F}_{\text{on boat by you}}$ and $\vec{F}_{\text{on you by boat}}$.

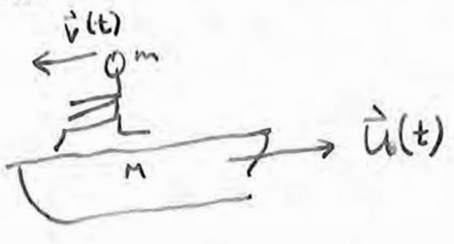
By Newton's 3rd law: $\vec{F}_{\text{on you by boat}} = -\vec{F}_{\text{on boat by you}}$.

So; $\frac{d\vec{P}_{total}}{dt} = 0$ at all times.

(Here, $\vec{P}_{total} = \vec{P}_{you} + \vec{P}_{boat}$)

And since $\vec{P}_{initial} = 0$ (No one is moving), we must have $\vec{P}_{total} = 0$ at all times

So:



$$\boxed{m\vec{v}(t) = -M\vec{u}(t)}$$

where $\vec{v}(t)$ = Velocity of person at time t

$\vec{u}(t)$ = Boat's velocity at time t

Note that boat must move to right to cancel out the person's momentum.

(B) When you stop walking, we have $\vec{V}(T)=0$
at $t=T$

$$m\vec{V}(T)=0$$
$$=-M\vec{u}(T) \Rightarrow \vec{u}(T)=0 \Rightarrow$$ Boat also stops moving at the same time as you.

(C) The boat has moved to the right by your walking to the left. When you stop, the boat stops too. Where is it when it stops?

The concept that connects the positions of individual objects in a system is the concept of center of mass.

Recall from the last lecture that the position of center of mass is the "weighted average" of the positions of the individual components of the system. (Here, they are the person & the boat.)

Namely,
$$X_{cm} = \frac{m X_{person} + M X_{boat}}{m+M}$$
Position of CM

At $t=0$ (initial time), no one is moving. So $\vec{P}_{total}(t=0)=0$

$$\Rightarrow \underbrace{(m+M)\vec{V}_{CM}(0)}=0$$

Here, we're using the fact that the $\boxed{\text{momentum of CM} = \vec{P}_{total}}$ (see ex 11 on pg 5-30)

so $\vec{V}_{cm}(t=0)=0$.

and since $\frac{d\vec{P}_{total}}{dt}=0$ at all times, $\vec{V}_{cm}(t)=0$ at all times (see (B).)

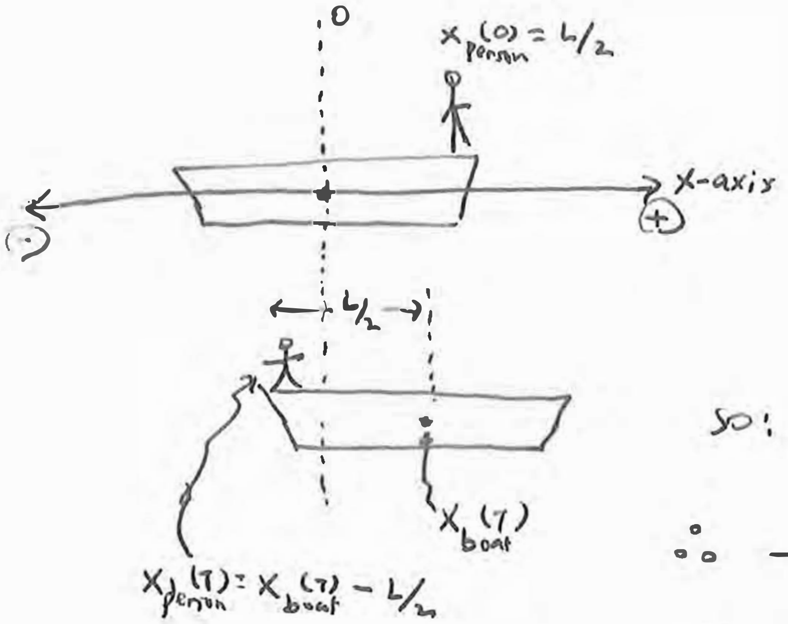
Thus, $X_{CM}(t) = X_{CM}(t=0)$

Never moves even though the boat & the person move.

So: $$X_{cm}(t) = 0 = \frac{m X_{person}(t) + M X_{boat}(t)}{m+M}$$

$$\Rightarrow \quad -m X_{person}(t) = M X_{boat}(t)$$

over



so: $x_{person}(T) = x_{boat}(T) - L/2$

$$\therefore \quad -m\,x_{person}(T) = M x_{boat}(T)$$

$$\Rightarrow \quad -m\left[x_{boat}(T) - L/2\right] = M x_{boat}(T)$$

$$\Rightarrow \quad \boxed{x_{boat}(T) = \frac{mL/2}{m+M}}$$

∎