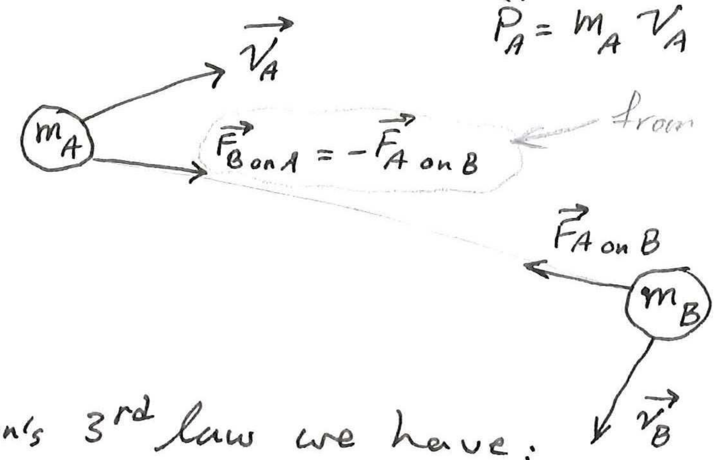
- Last week we saw that Newton's 2^{nd} law is :

$$\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt} \iff \vec{F}_{net\ ext} = \frac{d\vec{p}}{dt}$$

$$\vec{p} = m\vec{v}$$

- Let's derive the principle of "conservation of momentum":

$$\vec{P}_A = m_A \vec{v}_A$$



from Newton's 3^{rd} law

$$\vec{P}_B = m_B \vec{v}_B$$

From Newton's 3^{rd} law we have:

$$\vec{F}_{B\,on\,A} = -\vec{F}_{A\,on\,B} \longrightarrow \vec{F}_{A\,on\,B} + \vec{F}_{B\,on\,A} = 0$$

if external (outside) forces on A and B are negligible, then:

$$\vec{F}_{net\ on\ A} = \vec{F}_{B\,on\,A} + \text{negligible external forces} \approx \vec{F}_{B\,on\,A}$$

$$\vec{F}_{net\ on\ B} = \vec{F}_{A\,on\,B} + \text{neg. ext. forces} \approx \vec{F}_{A\,on\,B}$$

$$\frac{d\vec{P}_B}{dt} + \frac{d\vec{P}_A}{dt} = 0 \longrightarrow \frac{d}{dt}\left(\vec{P}_B + \vec{P}_A\right) = 0$$

we define $\vec{P}_{total} = \vec{P}_A + \vec{P}_B + \cdots$

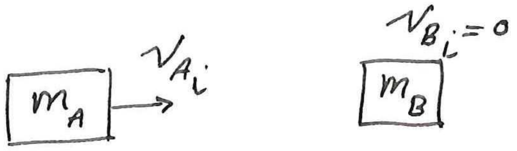
$$\Rightarrow \frac{d}{dt}\vec{P}_{tot} = 0 \Longrightarrow \vec{P}_{tot} = \text{constant vector in time}$$

* Therefore, if external forces are negligible, then the total momentum of a system is constant in time.

6-12-2016

Example 1

① before collision



② after collision

no ext. forces. Therefore:

$$\vec{P}_{tot} = \vec{P}_A + \vec{P}_B = m_A \vec{v}_A + m_B \vec{v}_B = \overrightarrow{\text{constan}} \text{ in time}$$

this means $\vec{P}_{tot_i} = \vec{P}_{tot_f}$

~~before~~ initial ~~after~~ final

Since we're dealing with vectors we need to define a coordinate system: $\longrightarrow x$

$$\vec{P}_{tot} = (P_x , P_y) \qquad P_x = m_A v_{A_x} + m_B v_{B_x} = \text{constant}$$

$$(m_A v_{A_x} + m_B \cancel{v_{B_x}})_{\text{initial}} = (m_A v_{A_x} + m_B v_{B_x})_{\text{final}}$$

0

$$\boxed{m_A v_{A_i} + 0 = m_A v_{A_f} + m_B v_{B\,\text{final}}}$$

We are going to talk about collisions.

~~But before that~~ Our friends in solving collision problems are conservation of energy and cons. of mom.

So we need to know how and when to use these laws.

Tip

- Steps for attacking problems using con. of mom and energy:
  1. Always split the problem into separate stages (with collisions treated individually)
  2. If ext. forces are negligible during a stage, then momentum is conserved during that stage.
     $$\vec{P}_{tot\ before} = \vec{P}_{tot\ after}$$
  3. If non-conservative forces (e.g. friction) are negligible during a stage, then mechanical energy is cons. during that stage
     $$(KE + PE)_{before} = (KE + PE)_{after}$$
  4. Solve (from here on it's just algebra)

Types of Collisions

total loss of KE ⟷ no loss of KE

totally inelastic collisions	inelastic (nearly elastic) collisons	totally elastic collisions
mom. is cons.	mom. is cons.	mom. is cons.

totally inelastic collisions

- two objects collide and stick together to form a single object.
- Example:
  - A bullet hitting a pendulum and staying in it
  - Two lumps of clay colliding
- kinetic energy is not entirely lost but this type of collision has the max. energy loss.

inelastic (nearly elastic) collisons

- tiny deformations (microscopically)
- Examples:
  - billiard balls

totally elastic collisions

- no deformations

~~No macroscopic object is involved in these collisions~~

- Examples:
  - scattering interacting (atomic)
  - 
  - Slingshot type grav. interactions between satellites and planets

totally inelastic collisions

$$\vec{P}_{tot_i} = \vec{P}_{tot_f}$$

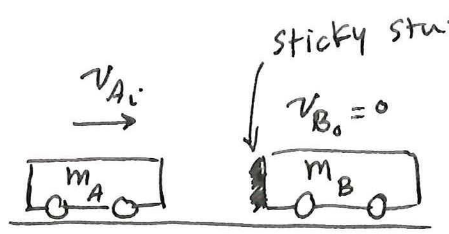
$$KE_i \neq KE_f$$

Elastic collisions (in your book) (inelastic + totally elastic)

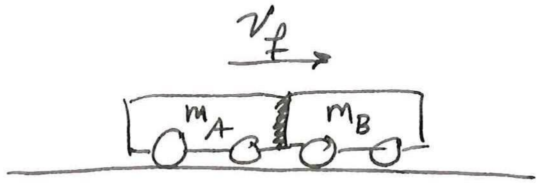
$$\vec{P}_{tot_i} = \vec{P}_{tot_f}$$

$$KE_i = KE_f$$

Example 2 (totally inelastic collision)



initial



final

mom. is cons. but energy is not!

$$\vec{P}_{tot_i} = \vec{P}_{tot_f}$$

$$\left(m_A v_{A_i} + \cancel{m_B v_{B_i}}^{\,0}\right) = \left(m_A v_{A_f} + m_B v_{B_f}\right)$$

$v_{A_f} = v_{B_f} = v_f$

$$m_A v_{A_i} = v_f\,(m_A + m_B) \Rightarrow v_f = \frac{m_A v_{A_i}}{m_A + m_B}$$

$$KE_i = \frac{1}{2} m_A v_{A_i}^2$$

$$KE_f = \frac{1}{2}(m_A + m_B)\left(\frac{m_A v_{A_i}}{m_A + m_B}\right)^2 = \frac{m_A^2 v_{A_i}^2}{2(m_A + m_B)} = \frac{1}{2} m_A v_{A_i}^2 \cdot \frac{m_A}{m_A + m_B}$$

$$\Delta KE = KE_f - KE_i = \frac{1}{2} m_A v_{A_i}^2 \left(1 - \frac{m_A}{m_A + m_B}\right) = \frac{1}{2} m_A v_{A_i}^2 \left(\frac{m_B}{m_A + m_B}\right)$$

$$= \frac{1}{2} m_A$$

Example 3 (Elastic Collisions)

both KE and P are conserved.

from example 1 we had:

$$m_A v_{A_i} = m_A v_{A_f} + m_B v_{B_f} \quad (a) \rightarrow m_A(v_{A_i} - v_{A_f}) = m_B v_{B_f} \quad (\mathrm{I})$$

From cons. of KE:

$$KE_i = KE_f \rightarrow KE_{A_i} + KE_{B_i} = KE_{A_f} + KE_{B_f}$$

$$\tfrac{1}{2} m_A v_{A_i}^2 + 0 = \tfrac{1}{2} m_A v_{A_f}^2 + \tfrac{1}{2} m_B v_{B_f}^2$$

$$\rightarrow m_A(v_{A_i}^2 - v_{A_f}^2) = m_B v_{B_f}^2$$

$$\rightarrow m_A(v_{A_i} - v_{A_f})(v_{A_i} + v_{A_f}) = m_B v_{B_f}^2 \quad (\mathrm{II})$$

(I) into (II):

$$\cancel{m_B} \cancel{v_{B_f}} (v_{A_i} + v_{A_f}) = \cancel{m_B} v_{B_f}^{\cancel{2}}$$

$$\rightarrow \boxed{v_{B_f} = v_{A_i} + v_{A_f}} \quad (\mathrm{III})$$

(III) into (a) for v_{B_f}

$$\cancel{m_A(v_{A_i} - v_{A_f}) =} \quad m_A v_{A_i} = m_A v_{A_f} + m_B(v_{A_i} + v_{A_f})$$

$$\boxed{v_{A_f} = \frac{v_{A_i}(m_A - m_B)}{m_A + m_B}}$$

III' $\rightarrow$ $v_{Af} = v_{Bf} - v_{Ai}$

III' into 0 for v_{Af}

$$m_A v_{Ai} = m_A (v_{Bf} - v_{Ai}) + m_B v_{Bf}$$

$$m_A v_{Ai} = m_A v_{Bf} + m_B v_{Bf} - m_A v_{Ai}$$

$$2 m_A v_{Ai} = v_{Bf} (m_A + m_B) \rightarrow \boxed{v_{Bf} = \frac{2 m_A v_{Ai}}{m_A + m_B}}$$

In fact, had we not assumed $v_{Bi} = 0$ then the most general equations would have been:

$$\begin{cases} v_{Af} = \dfrac{v_{Ai} (m_A - m_B)}{m_A + m_B} + \dfrac{v_{Bi} \cdot 2 m_B}{m_A + m_B} \\[2ex] v_{Bf} = \dfrac{v_{Bi} (m_B - m_A)}{m_A + m_B} + \dfrac{v_{Ai} \cdot 2 m_A}{m_A + m_B} \end{cases}$$

Let's see if our results make intuitive sense to us.

let's say $v_{B_i} = 0$:

we have 3 cases : $m_A \gg m_B$

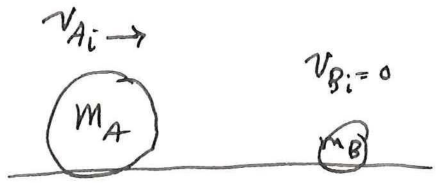
$m_A = m_B$

$m_A \ll m_B$

Case 1

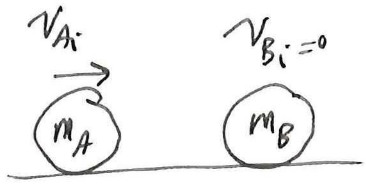
$m_A \rightarrow$ Bowling ball

$m_B \rightarrow$ ping pong ball



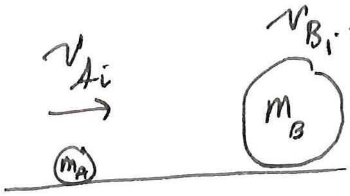
v_{A_f} is + $\qquad$ v_{B_f} is pos.

Case 2



v_{A_f} is 0 $\qquad$ v_{B_f} is pos.

Case 3



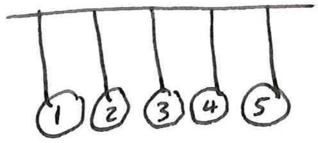
v_{A_f} is neg. $\qquad$ v_{B_f} is pos.
very small

---

no way v_{B_f} could be negative!

Newton's cradle

- Conservation of mom. and KE $\longrightarrow$



- Slows down due to energy loss because of tiny deformation (microscopic)

- This is identical to our example 3.

$$m_1 = m_2 = \cdots = m_5$$

$$v_{1f} = \frac{v_{1i}\,(m_1 - m_2)}{m_1 + m_2} = 0 \qquad v_{2f} = \frac{2 m_1 v_{1i}}{m_1 + m_2} = v_{1i}$$

$\vdots$

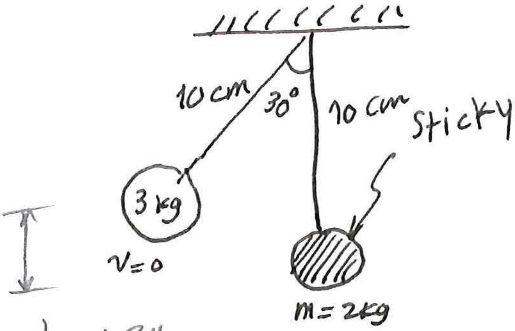
v_{2f} .... $v_{3f} = v_{1i}$

$\vdots$

$$v_{5f} = v_{1i}$$

Example 4

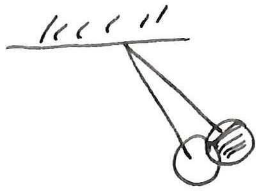
Recall the tip from earlier



$\Delta h = 1.34$

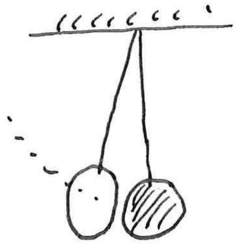
$10 - 10\cos 30$

3 stages



(pre collision)

1



$\rightarrow v = 0.512$ m/s

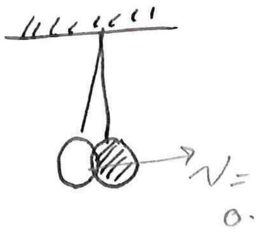
no collisions yet:

- only conservation of energy

$$KE_i + PE_i = KE_f + PE_f$$

- gravity → external force

2

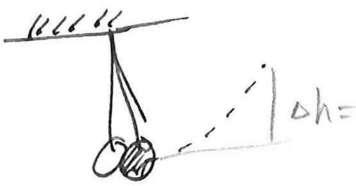


collision → totally inelastic

- mom. is conserved
- KE is not

(post collision)

3



no collision any more:

only cons of mechanical energy

$\Delta h = 0.48$ cm

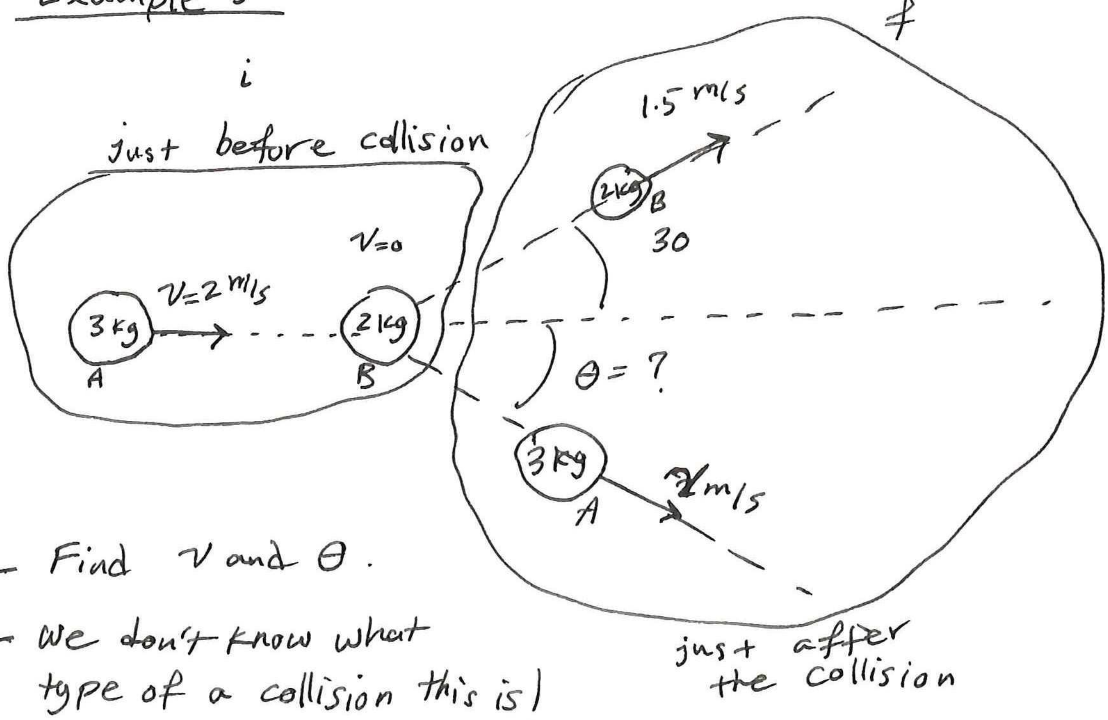
$$KE_i + PE_i = KE_f + PE_f$$

$\theta = 18°$

$h = 10\text{ cm} - 10\text{ cm}\cos\theta$

$10\cos\theta = 10 - 0.48$

Example 5



- Find v and θ.
- We don't know what type of a collision this is!

mom. is conserved in any collisions we deal with.

In x-dir

$$(m_A v_{Ax} + m_B v_{Bx})_i = (m_A v_{Ax} + m_B v_{Bx})_f$$

$$(3 \times 2 + 2 \times 0) = 3 \times v_{Ax_f} + 2(1.5 \underbrace{\cos 30}_{0.866})$$

$$v_{Ax_f} = 1.13 \text{ m/s}$$

In y-dir

$$(m_A v_{Ay} + m_B v_{By})_i = (m_A v_{Ay} + m_B v_{By})_f$$

$$(3 \times 0 + 2 \times 0) = 3 v_{Ay_f} + 2(1.5 \sin 30)$$

$$v_y = -0.5 \text{ m/s}$$

$$v = \sqrt{(-0.5)^2 + (1.13)^2} = 1.23 \text{ m/s}$$

$$\theta = \text{Arctan} \frac{|-0.5|}{1.13} = 23^\circ$$

in order to find out what type of collision this is we need to compare KE_i with KE_f :

$$\begin{cases} KE_i = \frac{1}{2} m_A v_{A_i}^2 = \frac{1}{2} 3 \times 2^2 = 6\ J \\ KE_f = \frac{1}{2} m_A v_{A_f}^2 + \frac{1}{2} m_B v_{B_f}^2 = \\ \quad = \frac{1}{2} 3 (1.23)^2 + \frac{1}{2} 2 (1.5)^2 = 4.5\ J \end{cases}$$

$$\Delta K = K_2 - K_1 = 4.5 - 6 = -1.5\ J$$

implies that energy was lost during collision

$\Downarrow$

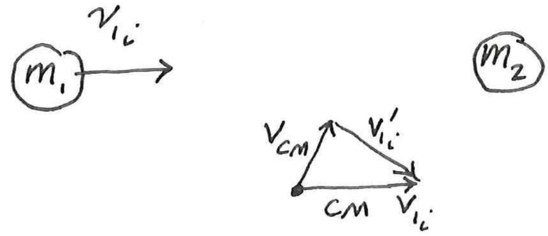
inelastic collisions.

Center of Mass frame (CM frame)

It's often easier to solve 2D collision problems if we choose CM frame as our frame of reference.

$\Leftarrow$ This is a ref. frame in which object 2 is at rest.

Say we want to switch to CM frame. This CM will have some velocity.



$$\vec{v}_{CM} + \underset{\substack{\uparrow \\ \text{velocity } v_{1_i} \text{ w.r.t CM}}}{\vec{v}_{1_i}'} = \vec{v}_{1_i}$$

$$\Rightarrow \begin{cases} \vec{v}_{1_i}' = \vec{v}_{1_i} - \vec{v}_{CM} \\ \vec{v}_{2_i}' = \vec{v}_{2_i} - \vec{v}_{CM} \end{cases}$$

from (9.5): $\vec{v}_{CM} = \dfrac{\sum_{j=1}^{2} m_j \vec{v}_j}{\sum_j m_j}$

$\Downarrow$ in our example

$$\vec{v}_{CM} = \frac{m_1 \vec{v}_{1_i} + 0}{m_1 + m_2}$$

$$\begin{cases} \vec{v}_{1_i}' = \vec{v}_{1_i} - \dfrac{m_1 \vec{v}_{1_i}}{m_1 + m_2} = \dfrac{m_2 \vec{v}_{1_i}}{m_1 + m_2} \\ \vec{v}_{2_i}' = \vec{v}_{2_i} - \vec{v}_{cm} = \dfrac{-m_1 \vec{v}_{1_i}}{m_1 + m_2} \end{cases}$$

Total mom. w.r.t CM is initially $\vec{P}_{tot_i} = m_1 \vec{v}_{1_i}' + m_2 \vec{v}_{2_i}' =$

$$= \frac{m_1 m_2 \vec{v}_{1i}}{m_1 + m_2} + \frac{-m_1 m_2 \vec{v}_{1i}}{m_1 + m_2} = 0$$

Thus, in CM frame total momentum of the system is zero before collision.

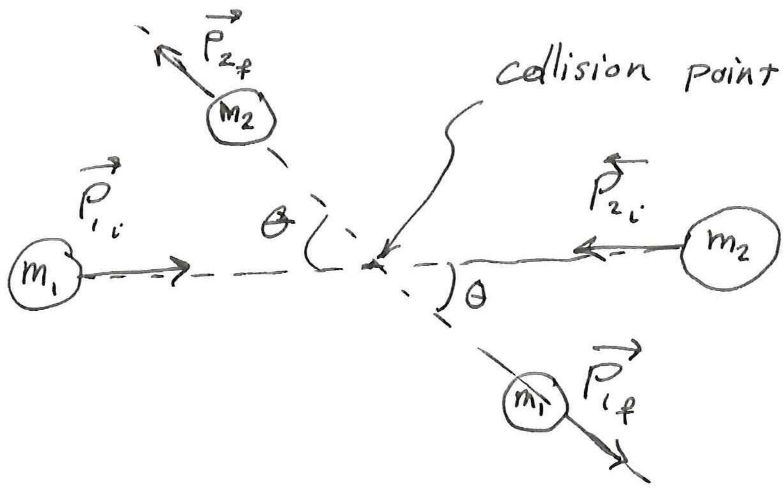
- Since collisions only involve internal forces that don't affect the CM, total mom. w.r.t. CM remains zero after the collision as well.

- From $\vec{P}_{tot} = \frac{m_1 m_2}{m_1+m_2}\vec{v}_{1_i} + \frac{-m_1 m_2}{m_1+m_2}\vec{v}_{1_i} = 0$

we see that in CM frame, both the initial and final momenta form pairs of oppositely directed vectors of equal magnitude

$$\left.\begin{aligned}\vec{P}_{tot_i} &= \vec{P}_{1_i} + \vec{P}_{2_i}\\ \vec{P}_{tot_f} &= \vec{P}_{1_f} + \vec{P}_{2_f}\end{aligned}\right\} \xrightarrow{\text{no ext. forces}} \vec{P}_{tot_i} = \vec{P}_{tot_f}$$

$$\vec{P}_{1_i} + \vec{P}_{2_i} = \vec{P}_{1_f} + \vec{P}_{2_f} = 0$$



if collision is elastic, it's further required that, by cons. of energy, ~~they~~ the initial and final speeds ~~of before and after~~ be the same. Therefore, angle θ describes the entire interaction