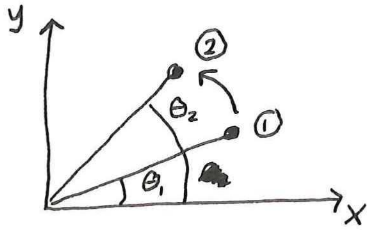
Rotational Kinematics

1. Angular Displacement



$\Delta\theta$ = angular displacement $= \theta_2 - \theta_1$

θ is measured counterclockwise from the x-axis in radians

ccw rotation $\rightarrow$ θ increasing

cw rotation $\rightarrow$ θ decreasing

most common convention

2. Angular velocity (average)

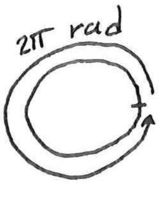
average velocity $= \dfrac{\text{displacement}}{\text{elapsed time}}$

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} \qquad \text{units: } \frac{\text{rad}}{s} \text{ or } \frac{1}{s}$$

omega

example: what's the earth's angular velocity in $\frac{\text{rad}}{s}$?

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{2\pi \text{ rad}}{1 \cancel{\text{day}}} \cdot \frac{1 \cancel{\text{day}}}{24 \cancel{\text{hr}}} \cdot \frac{1 \cancel{\text{hr}}}{3600 \text{ s}} \approx 7 \times 10^{-5} \frac{\text{rad}}{s}$$



3. Instantaneous angular velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad \text{units: } \frac{\text{rad}}{s} \text{ or } \frac{1}{s}$$

when θ increases with time (ccw rotation), then $\omega > 0$

" " decreases " " (cw rotation), then $\omega < 0$

~~useful~~

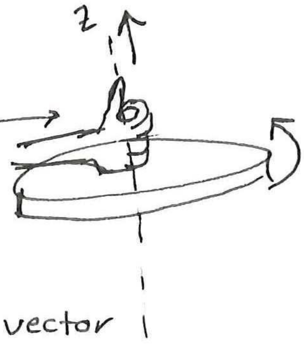
~~4. Angular speed~~

But, isn't angular velocity a vector? → yes

we need direction and magnitude.

Direction

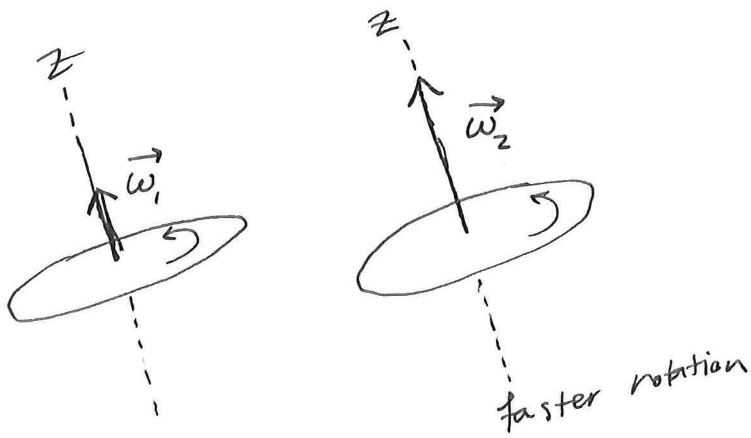
- CCW/CW are not proper directions for a vector
- we use the right-hand-rule to determine the direction of the angular velocity vector



- curl fingers in the direction of rotation
- your ~~thumb~~ shows the direction of ang. vel. vector

4. Magnitude (angular speed)

$\omega = |\vec{\omega}|$



$\vec{\omega}_1$ and $\vec{\omega}_2$ have the same directions

but $\omega_2 = |\vec{\omega}_2|$ is larger than $\omega_1 = |\vec{\omega}_1|$

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5. Average angular acceleration

$$\vec{\alpha} = \frac{\vec{\Delta\omega}}{\Delta t} \qquad \vec{\Delta\omega} = \vec{\omega}_2 - \vec{\omega}_1$$

alpha

units: $\frac{\text{rad}}{s^2}$ or $\frac{1}{s^2}$

6. Instantaneous ang. acc.

$$\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\vec{\Delta\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$$

7. Derive the two equations of angular (rotational) kinematics for constant angular acceleration:

$$\alpha = \frac{d\omega}{dt} = \text{constant} \qquad \omega = \frac{d\theta}{dt}$$

$$\alpha\, dt = d\omega \qquad \int \omega\, dt = \int d\theta$$

$$\int \alpha\, dt = \int d\omega \qquad \int_0^t (\omega_0 + \alpha t)\, dt = \int_{\theta_0}^{\theta} d\theta$$

$$\alpha \int_0^t dt = \int_{\omega_0}^{\omega} d\omega \qquad \omega_0 t + \alpha t^2 \frac{1}{2} = \theta - \theta_0$$

$$\cancel{\alpha(t_2 - t_1) = \omega - \omega_2}$$

$$\alpha\, t = \omega - \omega_0$$

$$\boxed{\omega = \omega_0 + \alpha t} \qquad \boxed{\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2}$$

$\omega_0 \equiv$ initial angular ~~velocity~~ speed at $t = 0$

$\theta_0 \equiv$ initial angle at $t = 0$

$\omega \equiv$ final ang. speed at time t

$\theta \equiv$ final angle at time t

Example

A wheel spinning at 100 RPM is brought to rest in 3 sec. How many turns does it make in the process?

$$\omega_0 = 100\ RPM = 100\ \frac{Rev}{min} = 100\ \frac{\cancel{Rev}}{\cancel{min}} \cdot \frac{2\pi\ rad}{1\ \cancel{Rev}} \cdot \frac{1\ \cancel{min}}{60\ s} = 10.47\ \frac{rad}{s}$$

~~we must~~ assuming constant acceleration α.

$$\omega = \omega_0 + \alpha t \longrightarrow 0 = 10.47 + \alpha\,(3\ sec)$$

$$\Rightarrow \alpha = -3.49\ rad/s^2$$

indicates that wheel is slowing down

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2$$

$$\theta = 0 + 10.47\,(3) + \tfrac{1}{2}(-3.49)(3)^2 = 15.7\ rad$$

$$\theta = 15.7\ rad = 15.7\ rad\ \frac{1\ rev}{2\pi\ rad} = \boxed{2.5\ \text{revolutions}}$$

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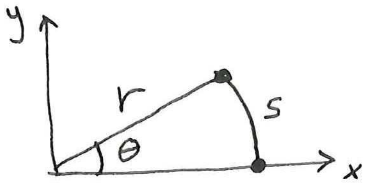
8. The arc distance travelled by a point on a rigid body:



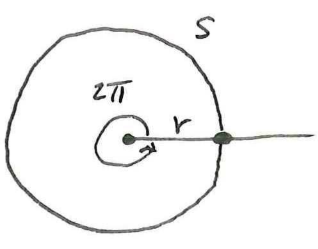
$s = r\theta$ ← angle (units radians)

radius (units: m)

Arc length = distance travelled along the circular path (units: m)

Check

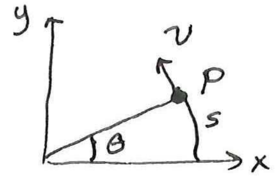
Full circle



$s = r2\pi$ = circumference

9. The speed of a point on a rotating rigid body.

tangential speed (linear)



$$v = \frac{ds}{dt} = \frac{\text{displacement along arc}}{\text{elapsed time}}$$

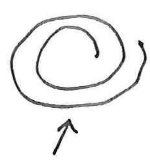
units : m/s

example

Find the speed of a point 20 cm from the axis of rotation, on a wheel rotating at 2 revolutions per second.

$$V = \frac{4\pi r}{1\ s} = \frac{4\pi (0.2\,m)}{1\ sec} \approx 2\ m/s$$

in 1 sec



$\theta = 4\pi$

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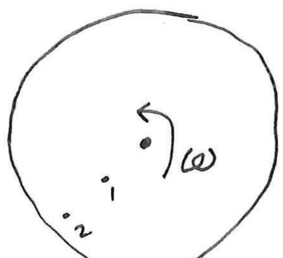
10. Relationship between angular speed and linear speed.

we have

$$S = r\theta$$

$$\frac{d}{dt}[S] = \frac{d}{dt}(r\theta) \rightarrow \frac{ds}{dt} = r\frac{d\theta}{dt} \rightarrow \boxed{V = r\omega}$$

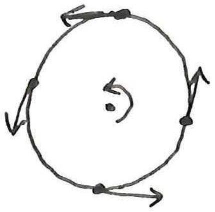


$V_2 > V_1$

why NASA launches rockets from Cape Canaveral?

and ESA launches from French Guiana?

- Acceleration is change in velocity over time.
  - change in velocity means change in its direction and/or change in its magnitude.



Is it possible for a point in ~~rotating~~ rotational motion to never experience acceleration?

No, because even if the angular speed is constant the direction of linear velocity of that point is constantly changing.

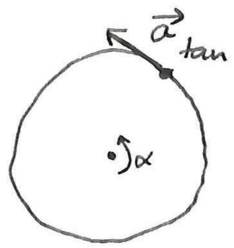
II. Relationship between the tangential and angular acceleration

$$v = r\omega \rightarrow \frac{dv}{dt} = \frac{d}{dt}(r\omega) \rightarrow \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$\downarrow$$

$$a_{tan} = r\alpha$$

$$\uparrow$$



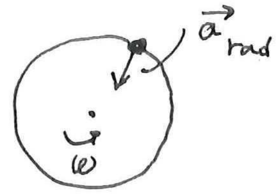
only nonzero if α is nonzero (ω is changing)

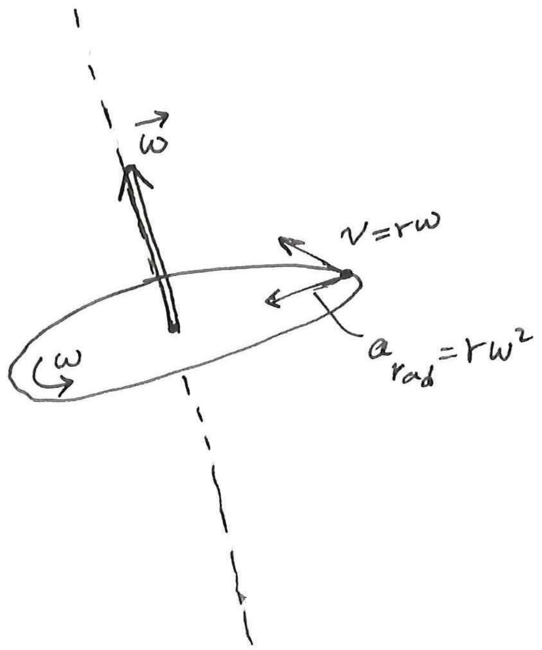
12. Relationship between radial acceleration (cetripetal acc) and angular speed.

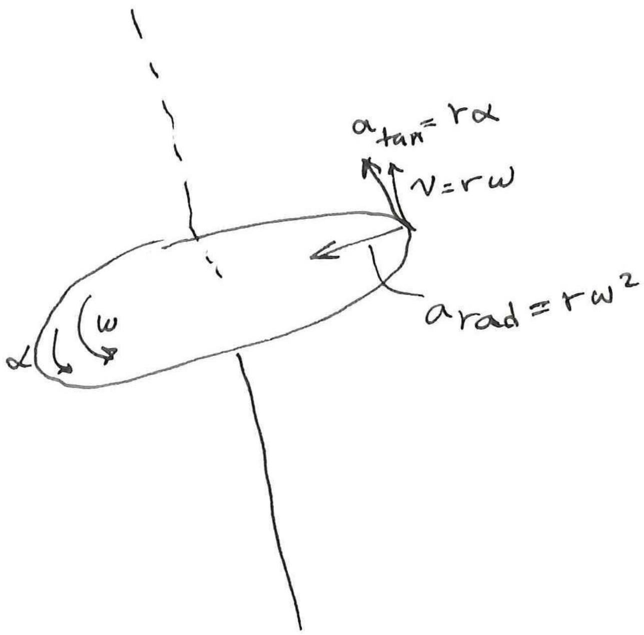
$$a_{rad} = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2$$

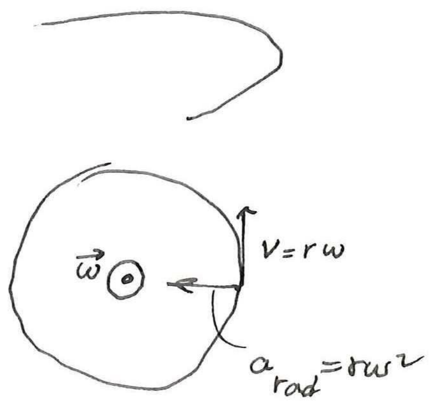
if $\omega \neq 0$ then $a_{rad} \neq 0$

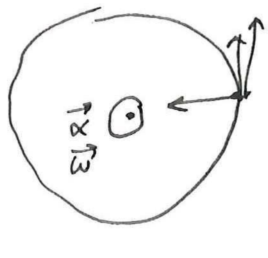
this is due to the constant change in the direction of the linear velocity.







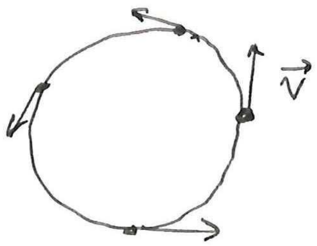




extra

a_{rad} is due to change in the direction of the vel. vector

$$a_{rad} = \frac{\text{change in velocity (direction)}}{\text{elapsed time}}$$



$\vec{v}$ has length $v = |\vec{v}|$

after one rev. change in direction of $\vec{v}$ is $2\pi v$

and elapsed time is $\frac{2\pi r}{v}$

$$a_{rad} = \frac{2\pi v}{\left(\frac{2\pi r}{v}\right)} = \frac{v^2}{r}$$

Kinematics equation

linear		rotational (angular)
$\bar{v} = \frac{1}{2}(v_0 + v)$	$x \leftrightarrow \theta$	$\bar{\omega} = \frac{1}{2}(\omega_0 + \omega)$
$v = v_0 + at$	$v \leftrightarrow \omega$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$a \leftrightarrow \alpha$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$		$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

$$v = r\omega$$

$$a_{tan} = r\alpha$$

$$a_{rad} = \frac{v^2}{r} = r\omega^2$$

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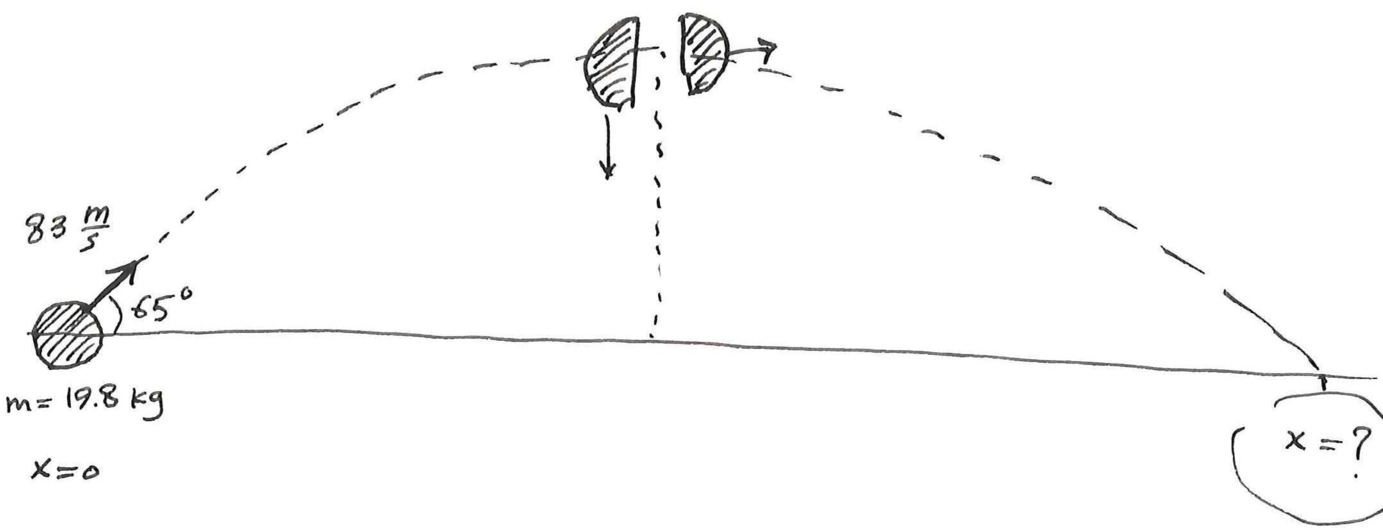
Problem

A rocket of mass 19.8 kg is launched with an initial speed of $83 \frac{m}{s}$ at 65° w.r.t. the ground.

The rocket splits into two pieces at the peak of its ~~height~~ flight. The first piece is stopped midair by a specially contrived explosion in such a way that its subsequent trajectory is straight down to the ground. How far will the second piece travel horizontally from its launchpad? Also calculate the kinetic energy of the whole rocket before and after the explosion.

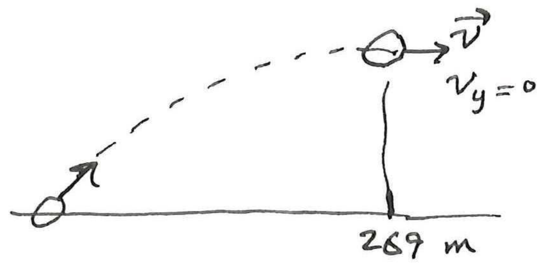


Solution

3 Stages :

Stage (I) Rocket rising
- initial event → launch
- final event → just before explosion

energy is cons. but momentum is not



$$v_y = v_{oy} + a_y t$$

$$0 = 83 \sin 65 - 9.8\, t \Rightarrow t = 7.68 \text{ s}$$

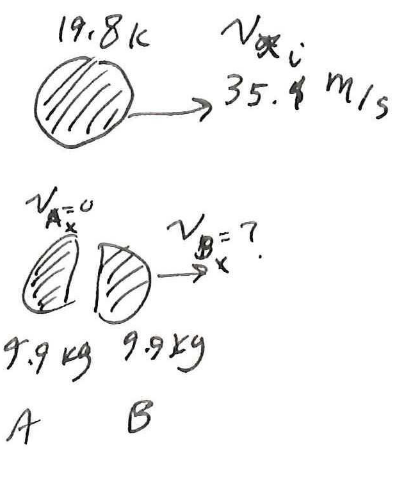
$$v_x = v_{ox} + a_x t$$

$$x = x_0 + v_{0x} t + \tfrac{1}{2} a_x t^2 \Rightarrow x = 0 + 83 \cos 65 (7.68) = 269 \text{ m}$$

$$v_x = 83 \cos 65 + 0 = 35.1 \text{ m/s}$$

Stage (II) Collision
- initial event: (max height is reached) right before explosion
- final event: just after exp.

mom. is cons. but energy is not because explosion adds energy to the system



$$m v_{x_i} = m_A v_{Ax} + m_B v_{Bx}$$

$$19.8 (35.1) = 9.9 (0) + 9.9\, v_{Bx}$$

$$\Rightarrow v_{Bx} = 70.2 \text{ m/s}$$

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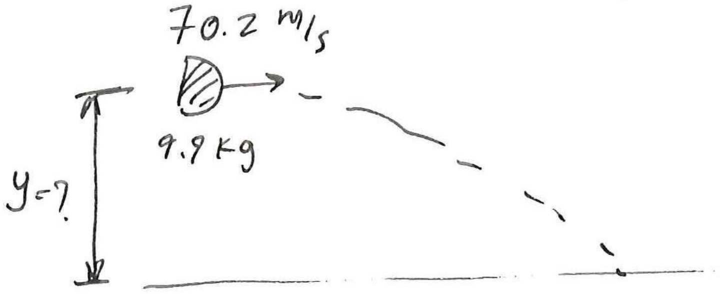
Stage (III) : Rocket falling $\begin{cases} i: \text{right after col.} \\ f: \text{before it hits the ground} \end{cases}$

energy is cons. / mom is not



from stage (I) $y = y_0 + v_{oy} t + \frac{1}{2} a_y t^2 =$

$= 0 + 83 \sin 65 (7.68) - \frac{1}{2}(9.8)(7.68)^2 =$

$= 288 \text{ m}$

$v_y = v_{oy} + a_y t$

$y = y_0 + v_{oy} t + 1/2\, a_y t^2$

$0 = 288 + 0 + 1/2(-9.8) t^2 \longrightarrow t = 7.68 \text{ s}$

$x = x_0 + v_{0x} t + 1/2\, a_x t^2 =$

$= 269 + 70.2(7.68) + 0 = \boxed{807 \text{ m}}$

$KE_{before} = 1/2\, m v^2 = 1/2\,(19.8)(35.1)^2 = 12196 \text{ J}$

$KE_{after} = 1/2\, m_A v_A^2 + 1/2\, m_B v_B^2 =$

$= 1/2(9.9)(0)^2 + 1/2(9.9)(70.2)^2 = 24393 \text{ J}$

$\Delta KE = 12197 \text{ J}$