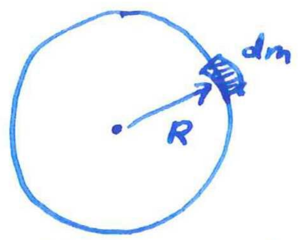
Solutions for the final exam – Physics 1A

Wed. February 1, 2017

Problem 1:

(a) $I_o = \int dm\, r^2$



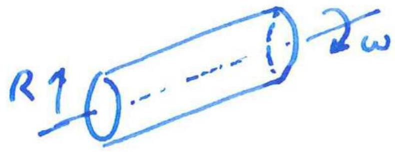
$= \int dm\, R^2$

$= R^2 \int dm$

$= R^2 M_o$

All mass element dm is located at same distance R from the center of the circle.

(b) $I_s = \int dm\, r^2$



$= \int dm\, R^2$

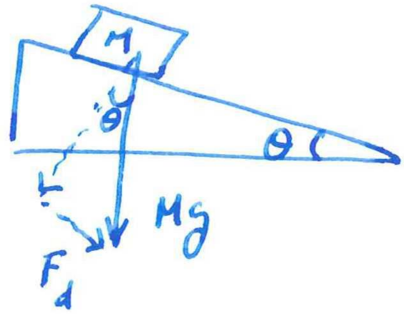
$= R^2 \int dm$

$= M_s R^2$

Like the onion ring, all mass element dm is at distance R from the axis of rotation.

(c) Say $M_s > M_o$.

No rolling, only sliding down the ramp.



$F_d = Mg\sin\theta$ $\quad F_d = Ma$

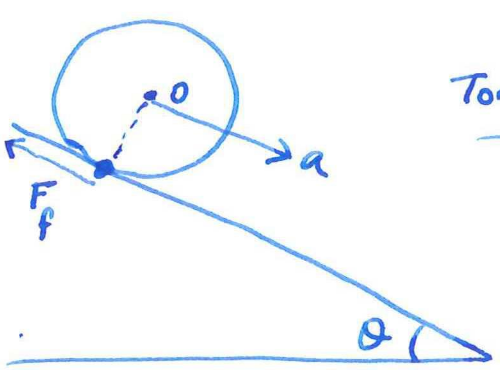
$\Rightarrow \boxed{a = g\sin\theta}$ ← not dependent on mass.

~~so both masses~~

So both the onion ring and the sausage have the same acceleration ⇒ Both reach the bottom of ramp at the same time

(d) Assume $M_s > M_o$.

Both roll without slipping.



Torque about center "o":

$$\tau = R F_f$$
$$\tau = I\alpha$$

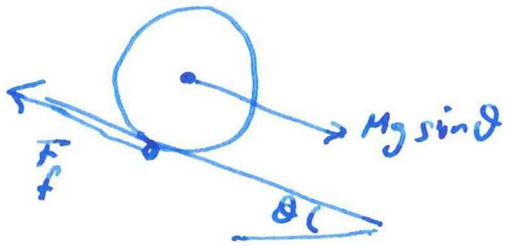
(Normal force doesn't affect the torque because

$\vec{r} \times \vec{F}_N = 0$.)

so $F_f = \dfrac{I\alpha}{R}$

linear motion (Newton's 2nd law):

$$Ma = Mg\sin\theta - F_f$$



Rolling without slipping $\Rightarrow R\alpha = a$

so: $MR\alpha = Mg\sin\theta - \dfrac{I\alpha}{R}$

$$\Rightarrow MR^2\alpha + I\alpha = MRg\sin\theta$$

$$\Rightarrow \alpha = \frac{MRg\sin\theta}{I + MR^2} \Rightarrow a = R\alpha = \frac{MR^2 g\sin\theta}{I + MR^2}$$

Thus, for sausage: $a_s = \dfrac{M_s R^2 g\sin\theta}{2M_s R^2} = \boxed{\dfrac{g\sin\theta}{2}}$

for onion ring: $a_o = \dfrac{M_o R^2 g\sin\theta}{2M_o R^2} = \boxed{\dfrac{g\sin\theta}{2}}$

$\Rightarrow$ Both reach the ramp bottom at the same time.

Problem 2

(a) $$m\frac{d^2x}{dt^2} = -k_1 x - k_2 x$$

$$\Rightarrow \boxed{m\frac{d^2x}{dt^2} + (k_1+k_2)x = 0}$$ ← equation of motion

(b) $$E_{tot.} = KE + PE$$

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{k_1+k_2}{2}x^2$$

(c) Amplitude $= A$.

At $x = \pm A$, $KE = 0$. $\Rightarrow E_{tot.} = \frac{k_1+k_2}{2}A^2$

so: energy conservation $\Rightarrow \frac{k_1+k_2}{2}A^2 = \frac{1}{2}m\left(V_{max}\right)^2$

↑ when $x=0$

$$\Rightarrow \boxed{\sqrt{\frac{k_1+k_2}{m}}\,A = V_{max}}$$

(d) $$\boxed{m\frac{d^2x}{dt^2} = -(k_1+k_2)x \pm \mu_k mg.}$$

↑ sign depends on direction of motion.

(**Note**: full points if student realizes this)

To get the actual sign, note that friction opposes direction of motion. $\Rightarrow F_f = -\mu_k mg\frac{\dot{x}}{|\dot{x}|}$.

↑ adjusts the sign.

so: $$\boxed{m\frac{d^2x}{dt^2} = -(k_1+k_2)x - \mu_k mg\frac{\dot{x}}{|\dot{x}|}}$$ ← Bonus +2 points if student gets to this point

(e) from the formula sheet: $x(t) = A\cos(\omega t + \phi)$

Going back to equation of motion in (a) and plugging $x(t)$ there, we get ω:

$$\omega^2 = \frac{k_1 + k_2}{m}$$

$$\Rightarrow \omega = \sqrt{\frac{k_1+k_2}{m}} \leftarrow \text{Angular frequency.}$$

$$2\pi \underbrace{f}_{\text{frequency}} = \omega$$

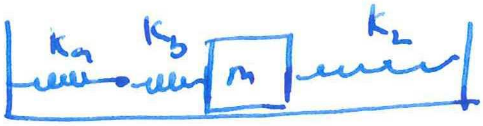
so: $f = \frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{m}}$.

so double mass: $m \to 2m$, $\quad f_{new} = \frac{1}{2\pi}\sqrt{\frac{k_1+k_2}{2m}}$

so: $$\frac{f_{new}}{f} = \frac{1}{\sqrt{2m}}\sqrt{m} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{f_{new} = \frac{1}{\sqrt{2}} f_{old}}$$

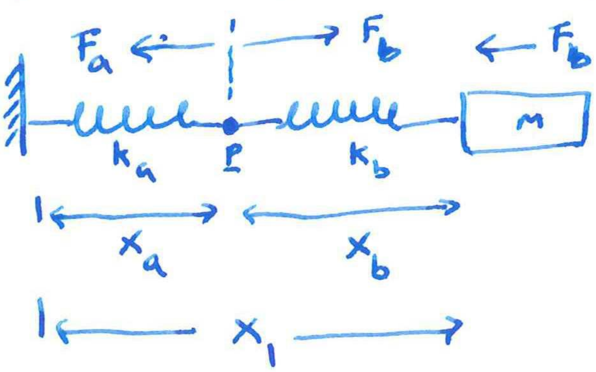
(f)



No friction.

k_a and k_b are massless springs.
so they cannot have net force on them.

First, find the effective spring constant of the new spring.
$\hookrightarrow k_{eff}$



At point P : $F_{net} = 0$.

so: $F_a = F_b$

$\Rightarrow k_a x_a = k_b x_b$ .... (1)

$x_a + x_b = x_1$ .... (2)

$k_{eff} x_1 = F_b = k_b x_b$ .... (3)

From (3) : $$\frac{k_{eff} x_1}{k_b} = x_b \quad \ldots (4)$$

Plugging into (2) : $$x_a = x_1 - x_b$$
$$= x_1 - \frac{k_{eff} x_1}{k_b}$$
$$= x_1 \left[\frac{k_b - k_{eff}}{k_b} \right] \quad \ldots (5)$$

Plugging (4) and (5) into (1), we get :

$$k_a \cancel{x_1} \left[\frac{k_b - k_{eff}}{k_b} \right] = \frac{\cancel{k_b}}{\cancel{k_b}} k_{eff} \cancel{x_1}$$

$$\Rightarrow \quad k_a k_b - k_a k_{eff} = k_{eff} k_b$$

$$\Rightarrow \quad k_{eff} = \frac{k_a k_b}{k_a + k_b}$$

So, equation of motion, like in (a), is :

$$m \frac{d^2 x}{dt^2} = -(k_{eff} + k_2) x$$

$$\Rightarrow \quad \omega = \sqrt{\frac{k_{eff} + k_2}{m}}$$

$$\Rightarrow \quad f = \frac{\omega}{2\pi}$$
$$= \boxed{\frac{1}{2\pi} \sqrt{\frac{\frac{k_a k_b}{k_a + k_b} + k_2}{m}}} \leftarrow \text{frequency}$$

Problem 3

(a) $A \to B$: $W_{AB} = -\int_A^B p \underbrace{dV}_{0}$

$= 0$

$B \to C$: $W_{BC} = -\int_A^B p\,dV$

$= -P_2 \int_{V_2}^{V_1} dV$

$= -P_2 (V_1 - V_2)$

$C \to A$: Isotherm at $T = T_0$ $\Rightarrow$ $pV = nRT_0$

$\Rightarrow$ $p = \frac{nRT_0}{V}$

So $W_{CA} = -\int_C^A p\,dV$

$= -nRT_0 \int_{V_1}^{V_2} \frac{dV}{V}$

$= -nRT_0 \ln\left(\frac{V_2}{V_1}\right)$

So: $W_{total} = W_{AB} + W_{BC} + W_{CA}$

$= \boxed{-P_2 (V_1 - V_2) - nRT_0 \ln\left(\frac{V_2}{V_1}\right)}$ ← total work done on gas.

(b) During constant volume process $A \to B$: $\Delta E = Q + \cancel{W}_0$

and $\Delta E = nC_V \Delta T$

so: $Q_{AB} = nC_V \Delta T$

$= nc_V (T_B - T_A)$ $\quad T_B = \frac{P_2 V_2}{nR}$ $\quad T_A = \frac{P_1 V_2}{nR}$

$= \frac{nc_V}{nR} (P_2 V_2 - P_1 V_2)$

$= \boxed{\frac{C_V V_2}{R} (P_2 - P_1)}$

$P_2 > P_1$ so $Q_{AB} > 0$

so heat enters box (gas gains heat)

OK to leave answer like this. But can also write as:

$Q_{AB} = \frac{C_V}{C_p - C_V} V_2 (P_2 - P_1)$

$= \frac{1/\gamma}{1 - 1/\gamma} V_2 (P_2 - P_1) \Rightarrow \boxed{Q_{AB} = \frac{V_2 (P_2 - P_1)}{\gamma - 1}}$

$\gamma > 1$.

(c) $W_{AC} = -W_{CA}$

$= nRT_0 \ln(V_2/V_1)$

$W_{CD} = 0$ ∵ no volume change

$W_{DA} = -\int_D^A p\,dV$

$= -P_1(V_2 - V_1)$

So $\boxed{W_{total} = nRT_0 \ln(V_2/V_1) - P_1(V_2 - V_1)}$

(d) $\underset{\underset{nC_V \Delta T}{\parallel}}{\Delta E_{C\to D}} = Q_{C\to D} + \cancel{W_{C\to D}}$ $\to 0$

$\Rightarrow Q_{C\to D} = nC_V(T_D - T_C)$

$= nC_V\left(\frac{P_1 V_1}{nR} - \frac{P_2 V_1}{nR}\right)$

$= \boxed{\frac{C_V V_1}{R}(P_1 - P_2)}$

$P_1 < P_2$

$\Rightarrow Q_{C\to D} < 0$

so heat exits box (gas loses heat)

ok to leave in this form.
but can also write as:

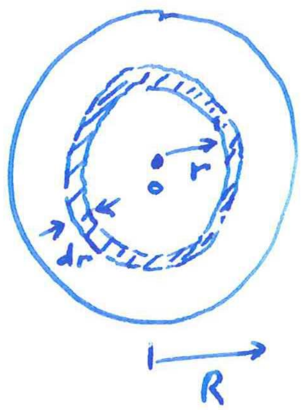
$Q_{C\to D} = \frac{C_V V_1}{C_p - C_V}(P_1 - P_2)$

$= \frac{1/\gamma \, V_1}{1 - 1/\gamma}(P_1 - P_2)$

$= \boxed{\frac{V_1}{\gamma - 1}(P_1 - P_2)}$ $(\gamma > 1)$

Problem 4

(a)



Break up into many infinitesimally thin (thickness = dr) concentric rings.

mass density $\rho = \dfrac{M}{\pi R^2}$

Thin ring has area $= 2\pi r\,dr$

so its mass is $dm = \rho\, 2\pi r\, dr$.

so $I = \int dm\, r^2$

$$= \int \rho\, 2\pi r \cdot r^2\, dr$$

$$= \rho\, 2\pi \int_0^R r^3\, dr$$

$$= \rho\, \cancel{2}\pi \frac{R^4}{\cancel{4}_2} \Big| = \frac{\cancel{\pi}}{2} R^4 \frac{M}{\cancel{\pi} R^2} = \boxed{\frac{M}{2} R^2}$$

(b) Total angular momentum of the system is conserved.

initial: $L_i = I_{bottom}\, \omega_1$

final: $L_f = I_{bottom}\, \omega_f + I_{top}\, \omega_f$

$$= \omega_f \left(I_{bottom} + I_{top} \right)$$

$\rightarrow L_i = L_f$

So: $\omega_f = \dfrac{I_{bottom}\, \omega_1}{I_{bottom} + I_{top}} \Rightarrow \boxed{\omega_f = \dfrac{m_1 R_1^2 \omega_1}{m_1 R_1^2 + m_2 R_2^2}}$

(c) $KE_{initial} = \dfrac{I_{bottom}\, \omega_1^2}{2} = \boxed{\dfrac{m_1 R_1^2}{4} \omega_1^2}$

$KE_{final} = \dfrac{I_{bottom} + I_{top}}{2} \omega_f^2 = \boxed{\dfrac{m_1 R_1^2 + m_2 R_2^2}{4} \omega_f^2}$

plugging into ω_f the result from (b), we get:

$$KE_{final} = \frac{1}{4} \frac{1}{m_1 R_1^2 + m_2 R_2^2} (m_1 R_1^2)^2 \omega_1^2$$

$$= \frac{1}{4} m_1 R_1^2 \omega_1^2 \frac{1}{1 + \frac{m_2}{m_1}\left(\frac{R_2}{R_1}\right)^2}$$

$$= KE_{initial} \cdot \underbrace{\frac{1}{1 + \frac{m_2}{m_1}\left(\frac{R_2}{R_1}\right)^2}}_{< 1.} \neq KE_{initial}.$$

so $KE_{final} < KE_{initial}$. due to heat generated during the collision of the 2 discs.

Problem 5

(a) $$\boxed{m \frac{d^2 y}{dt^2} = -ky.}$$ (no gravitational force here because we set $y=0$ to be the equilibrium position)

(b) for vertical oscillation,

$$\omega = \sqrt{\frac{k}{m}}.$$

for torsional oscillation: $I \frac{d^2\theta}{dt^2} + \gamma\theta = 0$

$$\Rightarrow \omega_{tor} = \sqrt{\frac{\gamma}{I}}$$

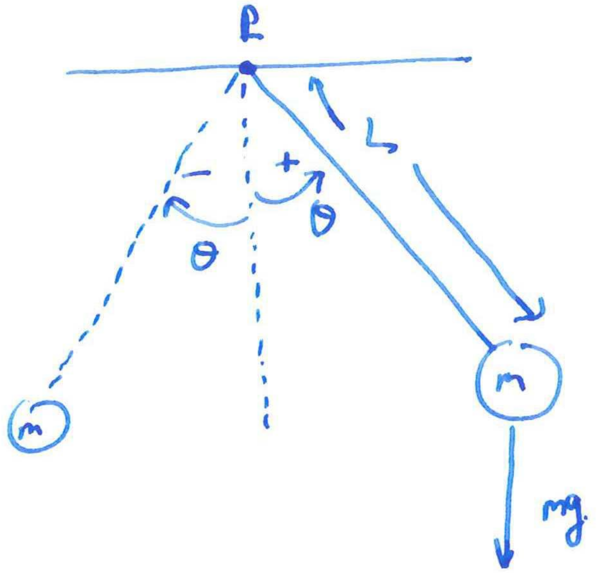
so, to have

$$\omega = \omega_{tor}$$

$$\Rightarrow \frac{k}{m} = \frac{\gamma}{I} \Rightarrow I = \frac{m\gamma}{k}$$

$$\Rightarrow \frac{\cancel{M}R^2}{2} = \frac{\cancel{M}\gamma}{k} \Rightarrow \boxed{R = \sqrt{\frac{2\gamma}{k}}}$$

(c)

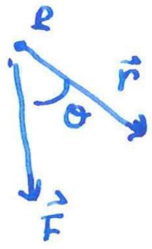


$\downarrow g$.

θ small. (i.e. $|\theta| \ll 1$) in radians.

Torque about pivot P is:

$$\tau = \vec{r} \times \vec{F}$$
$$= -Lmg\sin\theta$$



So: $I\alpha = \tau$

$$\Rightarrow I\frac{d^2\theta}{dt^2} = -Lmg\sin\theta \qquad I = mL^2$$

$$\Rightarrow \cancel{m}L^2\frac{d^2\theta}{dt^2} = -L\cancel{m}g\sin\theta$$

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$$\boxed{\omega = \sqrt{\frac{g}{L}}}$$

← m doesn't come ~~into~~ into equation so frequency does not depend on mass.

Note: Full points if student arrived at above result without using $\sin\theta \approx \theta$ approximation

If student used $\sin\theta \approx \theta$ to get $\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0}$,

and thus $\omega = \sqrt{\frac{g}{L}}$,

then $\boxed{+2 \text{ Bonus points}}$

Problem 6

$$h(x,t) = A\sin(kx+\omega t)$$

(a) wave moves to the left ($-x$ direction)

(b) λ = wavelength

$$k(x+\lambda) = kx + 2\pi \quad \text{by definition of wavelength.}$$

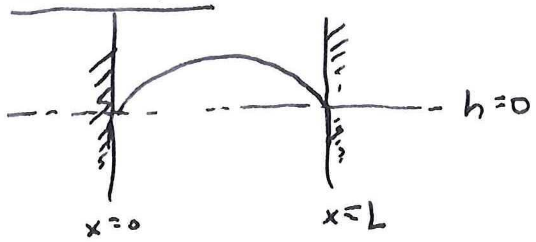
so: $k\lambda = 2\pi \Rightarrow \boxed{\lambda = \frac{2\pi}{k}}$

(c) The portion of the string at $x = x_0$ oscillates up and down as a simple harmonic oscillator with angular frequency ω. (and frequency $f = \frac{\omega}{2\pi}$)

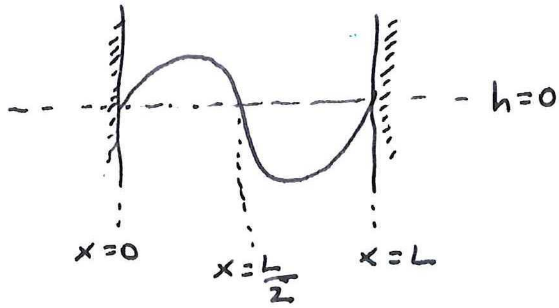
(d) For a transverse standing wave,

$$h(0) = h(L) = 0.$$

2 examples:



$\lambda_1 = 2L$



$\lambda_2 = L$